

The 3 D Yang–Mills system

Introductory Lecture Notes on the Schrödinger wave functional and Hamiltonian
 treatment by Karabali, Kim, Nair [1]

Field theory, when viewed as a medium coupled to a thermal bath, gains a useful new parameter : the temperature T . At the upper end of the T axis (g small) Yang–Mills theory (i.e. QCD without quarks, i.e. gluon black–body radiation) becomes susceptible to perturbation theory. The corresponding diagrammatics, however, runs against some “barrier of calculability”. At order g^6 for the pressure and at g^4 for the self-energy infinitely many diagrams (the “Linde sea”) contribute with the same order of magnitude. Does QCD exist? [2]. The Linde sea represents a physics by itself [3], namely that of a 3 D Yang–Mills system at zero temperature. Through learning about this system (non-perturbatively, but not necessarily exact), the barrier is overcome.

As usual, a lecture details the work done by others. In essence, there are only three others since, in the following, we shall focus on the paper [1]

D. Karabali, C. Kim and V. P. Nair, Nucl. Phys. B 524 (1998) 661

“Planar Yang–Mills theory : Hamiltonian, regulators and mass gap”

referred to as KKN for brevity. Part I of these notes was originally put in german: hep-ph/9908527. While translating, we step back from any reorganization. Let the original step by step understanding of this “very strange new matter” be a suitable low level introduction automatically. KKN’s § 2 is merely an *Outline of the main argument*. So, it has its preceeding papers [4]. The “paper after” [5] is examined in part II. Equations in [1] are referred to in the form [**n.m**]. But there might be no need for really looking into the original work.

One aspect of KKN’s treatment could be a bit fascinating, namely the “unification” of several areas of physics. The old idea of Feynman [6] in 1981 becomes explicit. The gauge orbit can be prepared. The (here Hermitean) Wess–Zumino–Witten action gets application, as does conformal field theory. Finally, the thermal field theory — a bit sickly after its euphoric phase around 1990 (Braaten–Pisarski resummation) — finds back to its original attitude of basically understanding reality. A circle gets closed.

¹ hschulz@itp.uni-hannover.de

Institut für Theoretische Physik, Universität Hannover, Appelstr. 2, D-30167 Hannover, Germany

Deser, Jackiw, Templeton, 1982 [7] :

The study of vector and tensor gauge theories in three-dimensional space-time is motivated by their connection to high temperature behavior of four-dimensional models, and is justified by the special properties which they enjoy.

Karabali, Kim, Nair, 1998 [1] :

... there is at least one interesting physical situation, viz., the high temperature phase of chromodynamics and associated magnetic screening effects, to which the (2+1)-dimensional theory can be directly applied.

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1 Two-dimensional classical electrodynamics

$$\operatorname{div} \vec{E} = \rho \quad (1.1) \quad \operatorname{div} \vec{B} = 0 \quad (1.3)$$

$$\operatorname{rot} \vec{E} = -\dot{\vec{B}} \quad (1.2) \quad \operatorname{rot} \vec{B} = \vec{j} + \dot{\vec{E}} \quad (1.4)$$

2D physics is a special case of 3D. Consider homogeneously charged straight lines (parallel to z-axis) to be the “point” charges. They form the densities $\rho(x, y, t)$ and $\vec{j} = (j_1(x, y, t), j_2(x, y, t), 0)$. Assuming that even $\operatorname{rot} \vec{B}$ attains the structure of \vec{j} , we have that

$$(1.1), (1.4) \text{ lead to } \vec{E} = (E_1(x, y, t), E_2(x, y, t), 0) .$$

$$\text{Now (1.2), (1.3) show that } \vec{B} = (0, 0, B(x, y, t))$$

has a third component only. This verifies the assumption. Maxwell has reduced to (1.1), (1.2), (1.4). No theorem is required to write $B = (\operatorname{rot} \vec{A})_3 = \partial_x A_2(x, y, t) - \partial_y A_1(x, y, t)$, hence introducing a 2-component vector potential. But we need (1.2) to allow for $\vec{E} = -\dot{\vec{A}} - \operatorname{grad} \phi$. The three fields E_1, E_2, B remain unchanged under regauging according to $\vec{A} \rightarrow \vec{A} + \nabla \chi(x, y, t)$ and $\phi \rightarrow \phi - \partial_t \chi(x, y, t)$.

The strict temporal gauge $\phi = 0$ (or Weyl gauge, or radiation gauge, [Muta, S.51]) does not fix completely. Without changing $\vec{E} = -\dot{\vec{A}}$ or $\vec{B} = \nabla \times \vec{A}$, we may still regauge by $\vec{A} \rightarrow \vec{A} + \nabla \chi(x, y)$. Note that χ must not depend on time.

Four-notation turns into “three-notation”, of course, with metrics $+- -$, $\mu = 0, 1, 2$, $(\phi, \vec{A}) =: A^\mu$ and $\partial^\mu = (\partial_0, -\nabla)$. Hence the connection between fields and potentials reads $E^j = -\partial^0 A^j + \partial^j A^0$, $B = -\partial^1 A^2 + \partial^2 A^1 = -\varepsilon_{jk} \partial^j A^k$ ($\varepsilon_{12} := 1$). As usual we define the field tensor $\partial^\mu A^\nu - \partial^\nu A^\mu =: F^{\mu\nu}$ and enjoy the resulting matrix version:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 \\ E_1 & 0 & -B \\ E_2 & B & 0 \end{pmatrix} , \quad F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & -B \\ -E_2 & B & 0 \end{pmatrix} . \quad (1.5)$$

Using this, we arrive at the Lagrangian

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{4} \operatorname{Tr} \left(\begin{matrix} \text{(1.5)} \\ \text{left} \\ \text{matrix} \end{matrix} \cdot \begin{matrix} \text{(1.5)} \\ \text{right} \\ \text{matrix} \end{matrix} \right) = \frac{1}{2} (\vec{E}^2 - B^2) \\ &= \frac{1}{2} \left([-\dot{\vec{A}} - \operatorname{grad} \phi]^2 - \frac{1}{2} [(\operatorname{rot} \vec{A})_3]^2 \right) . \end{aligned} \quad (1.6)$$

Now, when turning to the Hamiltonian density, the advantage of strict Weyl gauge $\phi = 0$ becomes obvious:

$$\mathcal{L} = \frac{1}{2} \dot{\vec{A}}^2 - \frac{1}{2} B^2 , \quad \vec{\Pi} = \dot{\vec{A}} = -\vec{E} \quad (1.7)$$

$$\mathcal{H} = \left[\dot{\vec{A}} \vec{\Pi} - \mathcal{L} \right]_{\text{eliminate } \dots} = \frac{1}{2} (\vec{\Pi}^2 + B^2) . \quad (1.8)$$

Only the two real fields A_1 and A_2 and their generalized momenta are left to work with.

2 Yang–Mills fields in 2+1 D

The specialities of non-abelian theory have nearly nothing to do with dimension. Just the Lorentz index μ now runs from 0 to 2. As if there were particles in the x-y plane (ψ with N colour components) too, we require invariance of any physics under \vec{r} - t dependent changes of the ψ phase. Changes of notation are psychological warfare. We therefore first remember our familiar Hannover notation. But let the coupling immediately be denoted by e (in place of g):

$$\begin{aligned}
 & U = e^{-ie\Lambda^a(x)T^a} \quad , \quad D_\mu = \partial_\mu - ieA_\mu^a T^a \\
 & \mathbf{A}_\mu := T^a A_\mu^a \quad , \quad \mathbf{A}_\mu \rightarrow \mathbf{A}_\mu^{[U]} = U \mathbf{A}_\mu U^{-1} - \frac{i}{e} U_{,\mu} U^{-1} \\
 & F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ef^{abc} A_\mu^b A_\nu^c \quad , \quad F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - ie [\mathbf{A}_\mu, \mathbf{A}_\nu] \\
 (1.5) : \quad & B^a = -F^{12a} = -(\partial^1 A^{2a} - \partial^2 A^{1a} + ef^{abc} A^{1b} A^{2c}) \\
 & E^{ja} = -F^{0ja} = -(\partial^0 A^{ja} - \partial^j A^{0a} + ef^{abc} A^{0b} A^{jc}) \quad , \quad A^{0a} \equiv 0 : E^{ja} = -\dot{A}^{ja} \\
 & \mathcal{L}_{\text{strictWeyl}} = -\frac{1}{4} F^{\mu\nu a} F_{\mu\nu}^a = \frac{1}{2} E^{ja} E^{ja} - \frac{1}{2} F^{12a} F^{12a} = \frac{1}{2} \dot{A}^{ja} \dot{A}^{ja} - \frac{1}{2} B^a B^a
 \end{aligned} \tag{2.1}$$

Among certain people around Chern and Simons (but nevertheless the article of Jackiw [8] in the Les Houches lectures of 1983 is very nice) it is common, however, to absorb the coupling e in the fields and to work with antihermitean field matrices. Then, the fields in (2.1) are processed as follows:

$$\begin{aligned}
 \Lambda^a & := e \Lambda^{\text{old}a} \quad , \quad A_j^a := e A_j^{\text{old}a} \quad , \quad F_{\mu\nu}^a := e F_{\mu\nu}^{\text{old}a} \quad , \\
 B^a & := -e B^{\text{old}a} \quad , \quad A_j := -ie \mathbf{A}_j^{\text{old}} \quad , \quad F_{\mu\nu} := -ie F_{\mu\nu}^{\text{old}} \quad .
 \end{aligned} \tag{2.2}$$

In parallel with these translations, we now (and forever) adopt the strict temporal gauge. Only the $2 * n$ fields $A_j^a(\vec{r})$ are left. They live in the plane $\vec{r} = (x, y)$. Let changes of gauge lie in a finite region : $\Lambda^a(\vec{r} \rightarrow \infty) \rightarrow 0$, and with (2.2) it is

$$U(\vec{r}) = \exp(-i\Lambda^a(\vec{r})T^a) \quad , \quad a = 1, \dots, N^2 - 1 =: n \quad . \tag{2.3}$$

KKN are not among those, who even prefer antihermitean generators. Hence, the following line, which was “forgotten” in (2.1), holds true old as new:

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} \quad , \quad [T^a, T^b] = if^{abc} T^c \quad . \tag{2.4}$$

By means of (2.2) the covariant derivative becomes nice. ∂_j is antihermitean, and this now harmonizes with the antihermiticity (and tracelessness) of the matrix fields:

$$D_j = \partial_j + A_j \quad , \quad A_j = -iT^a A_j^a \quad . \tag{2.5}$$

Their gauge transformation reads

$$A_j \rightarrow A_j^{[U]} = U A_j U^{-1} - U_{,j} U^{-1} \quad , \quad j = 1, 2 \quad . \tag{2.6}$$

Even field tensor and magnetic field loose ballast :

$$F_{jk}^a = \partial_j A_k^a - \partial_k A_j^a + f^{abc} A_j^b A_k^c \quad , \quad F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k] \quad , \quad (2.7)$$

$$B^a = \partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c \quad . \quad (2.8)$$

Finally, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2e^2} \dot{A}_j^a \dot{A}_j^a - \frac{1}{2e^2} B^a B^a =: \mathcal{T} - \mathcal{V} \quad . \quad (2.9)$$

May be, we are the first step in. (2.8) is found at KKN in the text below [2.4]. (2.6) is [2.1]. But (2.9) is not [2.4]. Well, perhaps something was wrong with the key board, setting e^2 in the numerator. (2.9) is fine, because it leads by

$$\Pi_j^a = \partial_{\dot{A}_j^a} \mathcal{L} = \frac{1}{e^2} \dot{A}_j^a \quad , \quad \mathcal{H} = \left[\Pi_j^a \dot{A}_j^a - \mathcal{L} \right]_{\text{eliminate } \dot{A}} = \frac{e^2}{2} \Pi_j^a \Pi_j^a + \mathcal{V} \quad (2.10)$$

to KKN's Hamiltonian density. Of course, the Lagrangian (2.9) can be shown to be invariant under the restricted U -transformations (2.3), (2.6), as usual.

3 Matrix parametrisation

The first important step into the KKN buisness needs only some rough philosophy. There are only the $2*n$ fields A_j^a . They regauge according to (2.6). But quantization must be restricted to physical fields (not connected through regauging). Hence, any better view into the space of fields A_j^a , any gain in harmony, would be fine.

The matrix parametrization could have been discovered as follows. We look at the gauge transformation (2.6) and play around with. For instance, we may place a unit matrix anywhere in this line. Let us put $1 = MM^{-1}$ in front of each U^{-1} ,

$$\begin{aligned} A_j^{[U]} &= U A_j M M^{-1} U^{-1} - U_{lj} M M^{-1} U^{-1} \\ &= U A_j M (UM)^{-1} - U_{lj} M (UM)^{-1} \quad , \end{aligned} \quad (3.1)$$

with the second line coming into mind for harmony. But something is not yet good with the last term. U likes M , and $(UM)_{lj}$ might appear there. Well, we may write $U_{lj} M = (UM)_{lj} - U M_{lj}$ and insert:

$$A_j^{[U]} = -(UM)_{lj} (UM)^{-1} + U [M_{lj} + A_j M] (UM)^{-1} \quad . \quad (3.2)$$

The first term is “harmonic”, and the second term we should like to get rid of. But before reaching this there is one more step towards harmony.

As the fields A_1^a and A_2^a are real, we may combine them to a general complex field $A^a := \frac{1}{2} (A_1^a + i A_2^a)$. Correspondingly, $A_1 = -i A_1^a T^a$ and $A_2 = -i A_2^a T^a$, which are antihermitean and traceless matrix fields, combine to

$$A := \frac{1}{2} \begin{pmatrix} A_1 + i A_2 \end{pmatrix} \quad . \quad (3.3)$$

This is one field. It is traceless but otherwise a general complex matrix. Given A , by preparing its hermitean and antihermitean part, one is led back uniquely to A_1 and A_2 . With (3.3), and with the definition

$$\partial := \frac{1}{2} \left(\partial_1 + i \partial_2 \right) \quad \left(\text{and } \bar{\partial} := \frac{1}{2} \left(\partial_1 - i \partial_2 \right) \text{ for later use} \right) , \quad (3.4)$$

we now turn back to (3.2) and combine these two equations (first one $+i$ the second) to a single one

$$A^{[U]} = -(\partial UM) (UM)^{-1} + U [\partial M + A M] (UM)^{-1} \quad (3.5)$$

with again the desire to get rid of the second term.

So far, nothing was assumed on M , except that it is a $N \times N$ matrix with an existing inverse. The square bracket in (3.5) vanishes, if for any given traceless field A there is a matrix field M such that

$$A = -(\partial M) M^{-1} . \quad (3.6)$$

If so, (3.5) becomes $A^{[U]} = -(\partial UM) (UM)^{-1}$ and tells us that regauging amounts to

$$M \rightarrow M^{[U]} = U M . \quad (3.7)$$

Moreover, (if so) we have that

$$(M^\dagger M)^{[U]} = (UM)^\dagger UM = M^\dagger U^\dagger U M = M^\dagger M =: H \quad (3.8)$$

is an invariant under gauge transformations.

To exhaust the above “if” we first realize intuitively that by running through the M space any matrix A is reached. However, A has to be traceless (but is unrestricted otherwise). Which way are the M 's to be restricted, correspondingly?

$$\underline{\text{The answer :}} \quad \det(M) \quad \text{must be a function of only } x + iy =: \bar{z} , \quad (3.9)$$

which is the only restriction on M .

$$\underline{\text{Proof :}} \quad 0 = \partial \ln(\det(M)) = \partial \text{Tr}(\ln(M)) = \text{Tr}((\partial M) M^{-1}) , \quad \text{q.e.d.} \quad (3.10)$$

One may also avoid using the \ln - \det formula. To reach a unique mapping from A space to M space, the function just mentioned in (3.9) can be even fixed:

$$\det(M) = 1 , \quad \text{i.e. } M \in \text{SL}(N, \mathbb{C}) . \quad (3.11)$$

In passing, (3.6) is [2.6], and (3.7) is [2.9].

Anything depending on x, y may be also understood to be a function of $z := x - iy$ and $\bar{z} := x + iy$, of course. For the (above) case that only one of these variables does occur, we have that

$$\begin{aligned} z := x - iy , \quad \partial := \frac{1}{2} (\partial_1 + i \partial_2) , \quad \partial f(z) = f'(z) , \quad \bar{\partial} f(z) = 0 \\ \bar{z} := x + iy , \quad \bar{\partial} := \frac{1}{2} (\partial_1 - i \partial_2) , \quad \bar{\partial} f(\bar{z}) = f'(\bar{z}) , \quad \partial f(\bar{z}) = 0 , \end{aligned} \quad (3.12)$$

i.e. differentiation with respect to the “wrong” variable gives zero.

4 Solving $A = -(\partial M)M^{-1}$ for M

Remember the way Born's approximation is derived: book down $(\Delta + k^2)\psi = V\psi$, consider the r.h.s. as a known inhomogeneity, solve for ψ by means of the Greens function of the Helmholtz operator $\Delta + k^2$ and then iterate by starting with a physical ψ . In the present case, the equation is $\partial M = -AM$, and the operator is ∂ .

To solve $\partial G(\vec{r}) = \delta(\vec{r})$ in 2D we write $G(\vec{r}) = 2(x - iy)f(r)$ to get

$$\partial G = (\partial_x + i\partial_y)(x - iy)f(r) = (2 + r\partial_r)f(r) = \frac{1}{r}\partial_r r^2 f(r) \stackrel{!}{=} \delta(\vec{r}) \quad . \quad (4.1)$$

The zero apart from the origin obviously needs that $f(r) \sim 1/r^2$. This function now must be embedded from the physical side, and the arising delta-function representation needs normalization:

$$\begin{aligned} f(r) &= \frac{\alpha}{r^2 + \varepsilon^2} \quad , \quad 1 \stackrel{!}{=} 2\pi \int_0^\infty dr r \left(\frac{1}{r} \partial_r r^2 \right) \frac{\alpha}{r^2 + \varepsilon^2} \quad \curvearrowright \quad \alpha = \frac{1}{2\pi} \\ G(\vec{r}) &= \frac{1}{\pi} \frac{x - iy}{r^2 + \varepsilon^2} = \frac{1}{\pi} \frac{z}{z\bar{z} + \varepsilon^2} \quad . \end{aligned} \quad (4.2)$$

Of course, $\varepsilon \rightarrow +0$ is meant. The limit may be only performed, $G \rightarrow 1/(\pi\bar{z})$, if the pitfall “ $\partial G = 0$ ” is anyhow excluded. Due to the translational invariance of ∂ we may write

$$\partial G(\vec{r} - \vec{r}') = \delta(\vec{r} - \vec{r}') \quad \curvearrowright \quad \partial \int d^2 r' G(\vec{r} - \vec{r}') (-A(\vec{r}')M(\vec{r}')) = -A(\vec{r})M(\vec{r}) \quad , \quad (4.3)$$

hence having obtained a special solution of the inhomogeneous equation $\partial M = -AM$:

$$M = M_{\text{hom}} - \int' GAM \quad , \quad \partial M_{\text{hom}} = 0 \quad , \quad M = 1 - \int' GAM \quad . \quad (4.4)$$

The fact, that the homogeneous equation is solved by any matrix $M_{\text{hom}}(\bar{z})$ is taken up again in § 11.1. But here, to the right in (4.4), we specify to $M_{\text{hom}} = 1$, as being one allowed choice to get a unique mapping.

To iterate (4.4), we now use a matrix language also with respect to space. Integrals are omitted (sum convention). M is a vector with the continuous index \vec{r} . Even the 1 in (4.4) is such a vector (having equal components). G is matrix, and $A(\vec{r}')$ may be replaced in (4.4) by the matrix $\mathbf{A}(\vec{r}', \vec{r}'') := A(\vec{r}')\delta(\vec{r}' - \vec{r}'')$. Let $\mathbf{1}$ stand for $\delta(\vec{r} - \vec{r}')$. The letter A to the very right in each term of the following equation is only a vector again. Herewith the iteration of (4.4) reads:

$$\begin{aligned} M &= 1 - GA + GA\mathbf{A}GA - GA\mathbf{A}GA\mathbf{A}GA + GA\mathbf{A}GA\mathbf{A}GA\mathbf{A}GA - \dots \\ &= 1 - \frac{1}{\mathbf{1} + G\mathbf{A}} GA = 1 - \frac{1}{1/G + \mathbf{A}} A = 1 - \frac{1}{\boldsymbol{\partial} + \mathbf{A}} A \quad , \end{aligned} \quad (4.5)$$

where we had read off $1/G = \boldsymbol{\partial}$ from $\boldsymbol{\partial} G = \mathbf{1}$. $\boldsymbol{\partial}$ is matrix (!), namely $\partial_{\vec{r}, \vec{r}'} = \frac{1}{2}\delta'(x - x') + \frac{i}{2}\delta'(y - y')$. Given a gauge field A , (4.5) tells us the corresponding member in the “underworld” of M 's. Eqs. [2.7], [2.8] are understood.

5 $M = V\rho$ — gauge invariant degrees of freedom

The unique mapping, we have reached in the preceding two sections, can be thought of in some analogy to the Fourier transformation. Any member of the space of gauge fields “knows” of its partner in underworld, i.e. in the space of $SL(N, \mathbb{C})$ matrices M , and vice versa :

$$M \xrightarrow{A = -(\partial M) M} A \quad , \quad A \xrightarrow{M = 1 - \frac{1}{\partial + A} A} M \quad . \quad (5.1)$$

To any physics among the A fields there is something corresponding going on among the M 's. This also applies to any manipulation as e.g. the preparing of the gauge orbit. Now, in the underworld, the gauge transformation was seen to be extraordinarily simple: $M \rightarrow M^{[U]} = UM$. So, this preparation and splitting off might be done there. Moreover, it reduces to a bit of thinking, if the following is true. Any $SL(N, \mathbb{C})$ matrix M can be uniquely written as the following *p r o d u c t*

$$\begin{aligned} M = V\rho \quad & \text{with} \quad VV^\dagger = 1, \det(V) = 1 \\ & \text{and} \quad \rho^\dagger = \rho, \det(\rho) = 1 \quad . \end{aligned} \quad (5.2)$$

Let the “bit of thinking” precede the proof. Given M , the term “uniquely” means, that the corresponding matrix ρ may be constructed. Many M 's have the same ρ . They differ by V . But V is a gauge transformation. M 's with the same ρ lie on the gauge orbit through ρ . Thats it. Splitting off the gauge orbit means reducing the space $SL(N, \mathbb{C})$ to the hermitean matrices ρ , or with the words of KKN below [2.9]: ρ represents the *gauge-invariant degrees of freedom*. — WOW!

As we shall see shortly, ρ is positive definit. Hence, in place of ρ , one can equivalently work with

$$\rho^2 = \rho^\dagger \rho = M^\dagger V V^\dagger M = M^\dagger M = H \quad , \quad \det(H) = 1 \quad . \quad (5.3)$$

H is the gauge invariant already noticed in (3.8). It is not hard to speculate that the Schrödinger wave functionals ψ of the 2+1D functional quantum mechanics must not depend on A or M but on only the physical degrees of freedom: $\psi(H)$. Don't we have already some very rough strategy? All we want to do can be formulated in the upper world. But for really doing it, the underworld is appropriate. With the mapping between the two worlds at hand, things will be managable anyhow. This strategy is more detailed in the next section and seen to be followed up through all the further headlines.

Having problems, it sometimes helps asking around. In the present case a few e-mails with York Schröder (DESY) led to the

Proof of $M = V\rho$:

1. $M^\dagger M$ is hermitean, hence can be diagonalized :

$$U M^\dagger M U^\dagger = \text{diag}(\lambda_1, \dots, \lambda_N) =: \text{diag} \quad . \quad (5.4)$$

2. As $M^\dagger M \varphi = \lambda \varphi \leadsto \int |M \varphi|^2 = \lambda$ shows, the diagonal elements λ_j are non-negative. They are even positive, because through

$$1 = \det(UM^\dagger MU^\dagger) = \det(\text{diag}) \quad (5.5)$$

zero-eigenvalues are forbidden.

3. We now define the hermitean matrix

$$\rho := U^\dagger \sqrt{\text{diag}} U \leadsto \det(\rho) = 1, \quad \frac{1}{\rho} = U^\dagger \frac{1}{\sqrt{\text{diag}}} U, \quad (5.6)$$

where $\sqrt{} := +\sqrt{}$. Given M , the matrix ρ is fixed uniquely, because, on one hand, $\rho^2 = U^\dagger \sqrt{\text{diag}} U U^\dagger \sqrt{\text{diag}} U = U^\dagger \text{diag} U = M^\dagger M$ due (5.4) and, on the other hand, the eigenvalues of ρ are all positive, since they are the elements of $\sqrt{\text{diag}}$. In passing, U is not fixed by M . Other than with a real rotation matrix, a diagonal matrix U_{ph} made up of phase factors can be split off from U to the left, and these factors recombine in $U_{\text{ph}}^\dagger \text{diag} U_{\text{ph}}$. U is not fixed, but $M^\dagger M$ and ρ are.

4. Once knowing that ρ has an inverse, we may solve $M =: V \rho$ for V and ask for its determinant and unitarity:

$$\begin{aligned} V &= M \frac{1}{\rho} \leadsto \det(V) = 1 \quad \text{and} \\ V^\dagger V &= \frac{1}{\rho} M^\dagger M \frac{1}{\rho} = U^\dagger \frac{1}{\sqrt{\text{diag}}} U M^\dagger M U^\dagger \frac{1}{\sqrt{\text{diag}}} U = 1. \end{aligned} \quad (5.7)$$

Somewhat, that can be written down, does exist, q.e.d. and thanks.

Initially we had some trouble to understand (5.2). Start with counting real parameters, Sergei Ketov said, to see whether $M = V \rho$ is possible at all. Both, Ketov and O. Lechtenfeld, referred to the relation to Lorentz transformations. Such counting is amusing, indeed. A complex $N \times N$ matrix has $2N^2$ elements, and the requirement $\det(M) \stackrel{!}{=} 1$ reduces to $2N^2 - 2 = 2n$. V is element of $\text{SU}(N)$ and has n real parameters: $V = e^{i\vec{\Lambda}\vec{T}}$. The more interesting factor is ρ with $\det(\rho) = 1$. Since it is positive definit, one may write

$$0 = \ln [\det(\rho)] = \text{Tr} [\ln(\rho)] \leadsto \ln(\rho) = \omega^a T^a, \quad \rho = e^{\vec{\omega}\vec{T}}. \quad (5.8)$$

Hence, ρ has n real parameters too, also found in the exponent. $2n = n + n$ — end of counting. From this point of view, $M = V \rho$ is nothing but a special way of booking down the elements of $\text{SL}(N, \mathbb{C})$.

6 The spaces | | | | | |---------------|--|-----------------|--| | \mathcal{A} | | \mathcal{C} | | | <hr/> | | | | | \mathcal{M} | | \mathcal{H} | | | | | \mathcal{G}_* | |

Here we relax a bit, to develop our state of mind and our strategy. The horizontal in the head line separates upper and underworld. All the five spaces, now provided with names,

are more or less known already :

\mathcal{A} : the space of all gauge fields (nevermind, whether we think in terms of the real $A_j^a(\vec{r})$ or its elegant combination to the complex traceless matrix field A).

\mathcal{M} : the space of all (\vec{r} dependent) matrices out of $SL(N, \mathbb{C})$.

\mathcal{H} : the physical subspace of the underworld = the space of all (\vec{r} dependent) hermitean $N \times N$ matrices H with determinant 1.

\mathcal{G}_* : the gauge group = the space of all (\vec{r} dependent) unitary matrices U = the elements of $SU(N)$.

\mathcal{C} : the space of only such A fields, which are not connected by gauge transformations = the space of gauge-invariant field configurations = **the** interesting physical subspace, in which quantization is allowed.

No grey hairs might grow by considering the following relation,

$$\text{space } \mathcal{C} = \frac{\text{space } \mathcal{A}}{\text{gauge group } \mathcal{G}_*} , \quad (6.1)$$

because (6.1) just defines the meaning of a quotient in the group chinese language. KKN give some amount of references for the geometry of the space \mathcal{C} (ref. [4] there). One could ask whether perhaps the spaces \mathcal{H} and \mathcal{C} are identical. Well, the H 's live in the underworld. If intergration over \mathcal{A} differs from integration over \mathcal{M} , because there is a Jacobian in between, then we expect something to be between \mathcal{H} and \mathcal{C} , too.

By the term ‘‘Schrödinger wave function’’ it is commonly made clear that ordinary ψ -function quantum mechanics is going on — instead of working with creators and annihilators. The latter are good for perturbation theory, but here we like to do it better. There is an other famous example for the uselessness of creators, namely the exact solutions by Bethe ansatz of a few special 1D many-particle systems as e.g. the Hubbard model. For a 1D oscillator we need an x -axis, the wave function attains complex values on. In field theory, each of the ∞ many points \vec{r} of the discretized space is the origin of a few (here $2 * n$) ‘‘field axes’’ (specific to \vec{r}). A point on the j - a -th axis gives the real value of A_j^a , and on these A -axes the wave function attains complex values: $\psi [A_j^a(\vec{r}) \text{ or } ?]$. Now, ‘‘Schrödinger wave-functional’’ is the appropriate term, indeed.

For a moment, let us forget about the gauge freedom. j, a, \vec{r} number variables. Correspondingly, a scalar product $\langle 1|2 \rangle$ between two states ψ contains $2 * n * \infty$ integrals:

$$\int \psi_1^* \psi_2 = \int dA_1^1(1) \dots dA_2^n(\infty) \Big|_? \psi_1^* [H(\vec{r})] \psi_2 [H(\vec{r})] =: \int d\mu(\mathcal{A}) \Big|_? \psi_1^* \psi_2 . \quad (6.2)$$

To the right in (6.2), the product of differentials has been given a name: volume element $d\mu(\mathcal{A})$ in the space \mathcal{A} .

The question marks in (6.2) refer to the problem. (6.2) makes sense only if before the integration has been restricted to physical variables. ψ depends on only these, and we

know them : $H(\vec{r})$. At this point, our rough strategy (“ask the underworld”) can be a bit detailed. Relate the measure $d\mu(\mathcal{A})$ with that (called $d\mu(\mathcal{M})$) of the underworld, i.e. obtain the Jacobian. Split off the gauge volume (in the underworld, of course, with best regards from Faddeev and Popov), and than turn back in upward direction :

$$\begin{array}{ccc} d\mu(\mathcal{A}) & & d\mu(\mathcal{C}) \\ \downarrow & & \uparrow \\ d\mu(\mathcal{M}) & \longrightarrow & d\mu(\mathcal{H}) \cdot \{d\mu(\mathcal{G}_*)\} \end{array} \quad (6.3)$$

In the next four sections we shall drown in details of measure and volume elements. But then, the ψ 's will call for a Hamiltonian. According to (2.9) and (2.10), and going to quantum mechanics by $\Pi \rightarrow -i\delta_A$, the kinetic energy will turn into a $2*n*\infty$ dimensional Nabla operator. This, in turn, when applied to $\psi(H)$, will become the Laplacian on \mathcal{C} .

A new volume element is expected to be the product of differentials of the new variables times a Jacobian. The latter is the absolute value of the “Jacobi matrix” $\mathfrak{S} := \partial(\text{old variables})/\partial(\text{new variables})$. For warming up, divide ds^2 in the second line by dt^2 and remember the kinetic energy, i.e. v^2 , of a particle. Obviously, spherical coordinates well illustrate that and how \mathfrak{S} is obtained from the metrics ds^2 (let them smile, who are experienced with general relativity: $g^{\mu\nu}$ from ds^2 and $\sqrt{\det(g)}$ in the action). Our starting point is the space \mathcal{A} . Its metrics in the third line is Euclidian and rather trivial (read $\int d^2r$ as $\sum_{\vec{r}}$). The other lines are outlook.

	space	elements	metrics	volume element
1	R^3	\vec{r}	$ds^2 = dx^2 + dy^2 + dz^2$	$d^3r = dx dy dz$
2	R^3	r, ϑ, φ $\mathfrak{S} := \frac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)}$	$ds^2 = dr^2 + r^2 d\vartheta^2 + r^2 \sin^2(\vartheta) d\varphi^2$ $ds^2 = \begin{pmatrix} dr \\ d\vartheta \\ d\varphi \end{pmatrix} \mathfrak{S}^T \mathfrak{S} \begin{pmatrix} dr \\ d\vartheta \\ d\varphi \end{pmatrix}$	$d^3r = dr r^2 d\vartheta \sin(\vartheta) d\varphi$ $d^3r = dr d\vartheta d\varphi \det(\mathfrak{S}) $
3	\mathcal{A}	A	$ds^2 = \int d^2r \delta A_j^a \delta A_j^a$	$d\mu(\mathcal{A}) = d\mu(\mathcal{M}) \det(D^\dagger D)$
4	\mathcal{M} $\text{SL}(N, \mathbb{C})$	M	$ds^2 = 8 \int d^2r \text{Tr} [(\delta M M^{-1})(M^{\dagger -1} \delta M^{\dagger})]$	$d\mu(\mathcal{M}) =$ $d\mu(\mathcal{H}) \cdot d\mu(\mathcal{G}_*)$
5	$\mathcal{H} = \frac{\text{SL}(N, \mathbb{C})}{\text{SU}(N)}$	H	$ds^2 = 2 \int d^2r \text{Tr}(H^{-1} \delta H H^{-1} \delta H)$	$d\mu(\mathcal{H})$
6	$\mathcal{G}_* = \text{SU}(N)$	U		$d\mu(\mathcal{G}_*)$
7	\mathcal{C}	A_{phys}		$d\mu(\mathcal{C}) = d\mu(\mathcal{H}) \det(D^\dagger D)$

Finally, let the variety of A 's be put in a scheme :

$$\begin{array}{ccccc} A_j = -iT^a A_j^a & \longleftarrow & 2n \text{ real } A_j^a & \longrightarrow & A^a = \frac{1}{2} (A_1^a + iA_2^a) \\ \downarrow & & & & \downarrow \\ \frac{1}{2} (A_1 + iA_2) & = & \text{one traceless } A & = & -iT^a A^a \end{array} \quad (6.5)$$

7 $d\mu(\mathcal{A}) \rightarrow d\mu(\mathcal{M})$: Jacobian determinant

At a point $\vec{r} = (x, y)$ in space, and at a position on its j - a -th field axis, let $\delta A_j^a(\vec{r})$ a small change of this position. Of course, this change affects each of the linear relations

(6.5) as well :

$$\begin{aligned}\delta A_1^a \delta A_1^a + \delta A_2^a \delta A_2^a &= (\delta A_1^a + i \delta A_2^a) (\delta A_1^a - i \delta A_2^a) = 4 \delta A^a \delta A^{a*} \\ &= 8 \operatorname{Tr} (T^a \delta A^a T^b \delta A^{b*}) = 8 \operatorname{Tr} (\delta A \delta A^\dagger) \quad .\end{aligned}\quad (7.1)$$

7.1 $ds_{\mathcal{A}}^2$ and δM

Certainly, using the relation $A = -(\partial M)M^{-1}$, the expression (7.1) can be written in terms of δM . Just

$$M M^{-1} = 1 \quad \curvearrowright \quad \delta M^{-1} = -M^{-1} \delta M M^{-1} \quad \text{and} \quad \partial M^{-1} = -M^{-1} (\partial M) M^{-1}$$

has to be used repeatedly :

$$\begin{aligned}\delta A &= -(\partial \delta M) M^{-1} - (\partial M) \delta M^{-1} \\ &= -\partial [\delta M M^{-1}] + \delta M \partial M^{-1} + (\partial M) M^{-1} \delta M M^{-1} \\ &= -\partial [\delta M M^{-1}] - \delta M M^{-1} (\partial M) M^{-1} + (\partial M) M^{-1} \delta M M^{-1} \\ &= -\{ \partial [\delta M M^{-1}] + [A, \delta M M^{-1}] \} \\ &= -\mathcal{D} \delta M M^{-1} \quad \text{with} \quad \mathcal{D} := \partial + [A, \quad]\end{aligned}\quad (7.2)$$

Probably, we could write d in place of δ , as well, but then parantheses might limit how far d acts. Let δ only refer to the quantity immediately following. The covariant derivative comes in several versions. In fundamental representation we have $D_j = \partial_j + A_j$ and may combine them to $D := \frac{1}{2}(D_1 + i D_2) = \partial + A$. \mathcal{D} in (7.2) is the commutator version in adjoint representation. Its index version D^{ab} comes into play when \mathcal{D} is applied to a matrix field $\Lambda^a T^a$:

$$\begin{aligned}\mathcal{D} \Lambda^a T^a &= T^a \partial \Lambda^a - i A^b [T^b, T^c] \Lambda^c \\ &= T^a D^{ac} \Lambda^c \quad \text{with} \quad D^{ac} := \delta^{ac} \partial + f^{abc} A^b \quad .\end{aligned}\quad (7.3)$$

A^b means $\frac{1}{2}(A_1^b + i A_2^b)$, of course. For (7.1) we also need δA^\dagger . Starting from $A^\dagger = -M^{\dagger -1} \bar{\partial} M^\dagger$ with $\bar{\partial} = \frac{1}{2}(\partial_1 - i \partial_2)$, every step of (7.2) appears daggered :

$$\delta A^\dagger = \dots = -\bar{\mathcal{D}} M^{\dagger -1} \delta M^\dagger \quad \text{with} \quad \bar{\mathcal{D}} := \bar{\partial} - [A^\dagger, \quad] \quad .\quad (7.4)$$

Inserting into (7.1) and summing by $\int d^2 r$ over space points the intermediate result is

$$ds_{\mathcal{A}}^2 = \int d^2 r \delta A_j^a \delta A_j^a = 8 \int d^2 r \operatorname{Tr} ([\mathcal{D} \delta M M^{-1}] [\bar{\mathcal{D}} M^{\dagger -1} \delta M^\dagger]) \quad , \quad (7.5)$$

This was just the first of three steps. In a second step (§ 7.2) we study δM and find — independently of (7.5) — the volume element $d\mu(\mathcal{M})$. It needs a third step (§ 7.3) to establish the relation between the measures $d\mu(\mathcal{A})$ and $d\mu(\mathcal{M})$.

7.2 $d\mu(\mathcal{M})$

Elements of a group (here M 's) are related by multiplication. To study the metrics, N. Dragon said, one starts from the 1-element. An infinitesimal deviation from 1 can be parametrized by $1 + \frac{1}{2}\vec{\varepsilon} \vec{T}$. The components ε^a of $\vec{\varepsilon}$ are complex. Linear in $\vec{\varepsilon}$ $\det(M) = 1$ is guaranteed. Now we settle down in the middle of the group and like to formulate a small difference between M and its neighbour $M + \delta M$ — by multiplication :

$$\begin{aligned} M + \delta M &= (1 + \frac{1}{2}\vec{\varepsilon} \vec{T}) M \quad \leadsto \quad \delta M = \frac{1}{2}\vec{\varepsilon} \vec{T} M \\ \delta M M^{-1} &= \frac{1}{2}\vec{\varepsilon} \vec{T} \quad , \quad \varepsilon^a = 4 \operatorname{Tr} (T^a \delta M M^{-1}) \quad . \end{aligned} \quad (7.6)$$

The prefactor $\frac{1}{2}$, seemingly unnecessary, will keep the results simple. $d\mu(\mathcal{M})$ follows from the metrics, and the latter needs quadratic infinitesimal quantities. Already the first tempting idea works well :

$$8 \operatorname{Tr} (\delta M M^{-1} M^{\dagger -1} \delta M^{\dagger}) = 2 \operatorname{Tr} (\varepsilon^a T^a \varepsilon^{b*} T^b) = \varepsilon^a \varepsilon^{a*} = \varepsilon_1^a \varepsilon_1^a + \varepsilon_2^a \varepsilon_2^a \quad , \quad (7.7)$$

where of course a is summed over, and $\varepsilon_1^a := \Re(\varepsilon^a)$, $\varepsilon_2^a := \Im(\varepsilon^a)$. Hence, the metrics in the space of \vec{r} -dependent fields M is

$$ds_{\mathcal{M}}^2 = 8 \int d^2 r \operatorname{Tr} (\delta M M^{-1} M^{\dagger -1} \delta M^{\dagger}) = \int d^2 r \varepsilon^a \varepsilon^{a*} \quad . \quad (7.8)$$

The left half of this line is [2.12] and the fourth line in the table (6.4), the right half is something own. By construction, the metrics is kartesian. So, in our ε -language,

$$d\mu(\mathcal{M}) = \prod_{\vec{r}} \prod_{a,j} \varepsilon_j^a = \prod_{\vec{r}} \prod_a \varepsilon_1^a \varepsilon_2^a \quad (7.9)$$

is the volume element (the *Haar measure*) in the space \mathcal{M} . (7.9) is the starting point for splitting off the volume in § 8.

7.3 $d\mu(\mathcal{A}) = d\mu(\mathcal{M})$ times Jacobian

Only now a big Jacobian may be announced. We start from the metrics (7.5), express δM through (7.6) by ε 's there and make use of (7.3), (7.4) :

$$\begin{aligned} ds_{\mathcal{A}}^2 &= 2 \int d^2 r \operatorname{Tr} ([\mathcal{D} \varepsilon^a T^a] [\overline{\mathcal{D}} \varepsilon^{b*} T^b]) = 2 \int d^2 r \operatorname{Tr} ([T^a D^{ac} \varepsilon^c] [T^d D^{db*} \varepsilon^{b*}]) \\ &= \int d^2 r [D^{ac} \varepsilon^c] [D^{ab*} \varepsilon^{b*}] \quad \text{where} \quad D^{ab*} := \{ \delta^{ab} \overline{\partial} + A^{c*} f^{acb} \} \\ &= \int d^2 r [D \vec{\varepsilon}] \cdot [D \vec{\varepsilon}]^* = \int d^2 r ([\Re(D \vec{\varepsilon})]^2 + [\Im(D \vec{\varepsilon})]^2) \\ &\quad D =: D_1 + i D_2 \quad , \quad \vec{\varepsilon} = \vec{\varepsilon}_1 + i \vec{\varepsilon}_2 \quad : \\ &= \int d^2 r ([D_1 \vec{\varepsilon}_1]^2 + [D_2 \vec{\varepsilon}_2]^2 - 2 [D_1 \vec{\varepsilon}_1] [D_2 \vec{\varepsilon}_2] \\ &\quad + [D_1 \vec{\varepsilon}_2]^2 + [D_2 \vec{\varepsilon}_1]^2 + 2 [D_1 \vec{\varepsilon}_2] [D_2 \vec{\varepsilon}_1]) \\ &= \int d^2 r \left[\begin{pmatrix} D_1 & -D_2 \\ D_2 & D_1 \end{pmatrix} \begin{pmatrix} \vec{\varepsilon}_1 \\ \vec{\varepsilon}_2 \end{pmatrix} \right] \cdot \left[\begin{pmatrix} D_1 & -D_2 \\ D_2 & D_1 \end{pmatrix} \begin{pmatrix} \vec{\varepsilon}_1 \\ \vec{\varepsilon}_2 \end{pmatrix} \right] =: [\vec{\varepsilon}] \cdot [\vec{\varepsilon}] \quad . \end{aligned} \quad (7.10)$$

In the third line “ D ” is shorthand for D^{ab} of course (sorry, too many D ’s, simply omit the former meaning $\partial + A$ now). The operator D is not only a $n \times n$ matrix, but also contains differentiation. In (7.10), their action is limited by square brackets.

Turning to the expression $\vec{[]} \cdot \vec{[]}$ to the right in the last line of (7.10), the integral $\int d^2 r$ was omitted by extending the sum convention. At the same moment $\vec{[]}$ must be viewed as a big \vec{r} -indexed vector. Correspondingly, the operators D become big matrices \mathbf{D} , carrying the index pair \vec{r}, \vec{r}' . \mathbf{D} acts on ε by $\int' D_{rr'} \varepsilon_{r'}$. We encountered this language already in (4.5) and therefore maintain the boldface-Notation. These changes of mind are required for correctly reading off the Jacobi-matrix \mathfrak{S} from (7.10). The \mathcal{A} -space volume element is now obtained as

$$d\mu(\mathcal{A}) = d\mu(\mathcal{M}) |\det(\mathfrak{S})| = d\mu(\mathcal{M}) \left| \det \begin{pmatrix} \mathbf{D}_1 & -\mathbf{D}_2 \\ \mathbf{D}_2 & \mathbf{D}_1 \end{pmatrix} \right|. \quad (7.11)$$

The matrix \mathfrak{S} is real. But $\det(\mathfrak{S})$ has not yet the desired form $\det(\mathbf{D}^\dagger \mathbf{D})$. The latter is achieved by the following nice derivation (diagonalize \mathcal{I} , J. Schulze said):

$$\begin{aligned} \mathfrak{S} &= \begin{pmatrix} \mathbf{D}_1 & -\mathbf{D}_2 \\ \mathbf{D}_2 & \mathbf{D}_1 \end{pmatrix} = \mathbf{D}_1 \mathbf{1} + \mathbf{D}_2 \mathcal{I}, \quad \mathcal{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ W &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad W \mathcal{I} W^\dagger = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \det(W) = \det(W^\dagger) = 1, \\ \det(\mathfrak{S}) &= \det(W \mathfrak{S} W^\dagger) = \det \begin{pmatrix} \mathbf{D}_1 + i\mathbf{D}_2 & 0 \\ 0 & \mathbf{D}_1 - i\mathbf{D}_2 \end{pmatrix} = \det(\mathbf{D} \mathbf{D}^*) \quad (7.12) \end{aligned}$$

The \mathcal{A} -space volume-element now reads $d\mu(\mathcal{A}) = d\mu(\mathcal{M}) |\det(\mathbf{D} \mathbf{D}^*)|$. Fine? Is there a need for those absolute value bars?

To the hell with all the errors ever produced in physics papers by missing absolute value bars around Jacobians. Remembering (7.3) let us book down the matrix \mathbf{D} with all indices,

$$\mathbf{D} : \quad D_{rr'}^{ab} = \delta^{ab} \partial_{rr'} + A^c(\vec{r}) f^{acb} \delta_{rr'}, \quad (7.13)$$

and notice that

$$\begin{aligned} \mathbf{D}^T : \quad (D^T)_{rr'}^{ab} &= \delta^{ab} (-\partial_{rr'}) + A^c(\vec{r}) f^{bca} \delta_{rr'} \\ \curvearrowright \quad \mathbf{D}^T &= -\mathbf{D}, \quad \text{hence} \quad \mathbf{D}^* = -\mathbf{D}^\dagger, \end{aligned} \quad (7.14)$$

because $A^c = \frac{1}{2}(A_1^c + iA_2^c)$, A_j^c real, and $\partial_{rr'} = -\partial_{r'r}$, see the line below (4.5). Vector arrows over r -indices are suppressed for simplicity. But, please, add them by mind. If the hermitean matrix $\mathbf{D}^\dagger \mathbf{D}$ has no zero eigenvalue (we endure the risk), then we may state that

$$-\mathbf{D} \mathbf{D}^* = \mathbf{D} \mathbf{D}^\dagger \stackrel{\text{is positive definit}}{\curvearrowright} |\det(\mathbf{D} \mathbf{D}^*)| = \det(\mathbf{D}^\dagger \mathbf{D}) =: e^\Gamma, \quad (7.15)$$

and

$$d\mu(\mathcal{A}) = d\mu(\mathcal{M}) \det(\mathbf{D}^\dagger \mathbf{D}) \quad (7.16)$$

is the result of this section 7. It completes line 3 in the table (6.4). The quantity Γ defined in (7.15) is real. Aside, through (7.10) to (7.15) we proceeded by two lines text in KKN below [2.12].

8 $d\mu(\mathcal{M}) \rightarrow d\mu(\mathcal{H})$: splitting off the “volume”

To the present job of making path for pedestrians, it was very helpful to have Jens Reinbach being engaged too, in particular in this section and the next. But do not ask “who–what”, because all has been actually done by KKN.

The “separation ansatz” for decoupling unphysical degrees of freedom is known from § 5 to be

$$M = U \rho \quad \leadsto \quad \delta M = \delta U \rho + U \delta \rho \quad . \quad (8.1)$$

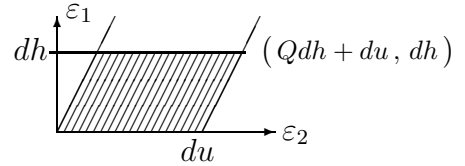
Now preferring U rather than V signals a slight change in philosophy: all possible gauge transformations U are started from a physical “point” ρ . As the content of this section can be well stated in words, we do something strange, start with the summary (next paragraph up to the figure) and go into the details only afterwards.

On one hand, we know the \mathcal{M} –space measure $d\mu(\mathcal{M}) = \prod_{\vec{r}} \prod_a \varepsilon_1^a \varepsilon_2^a$ from (7.9), and the relation of ε ’s with δM from (7.6), on the other. Through the split (8.1) it will turn out that ε_1^a is a pure ρ –ic expression (depending on only ρ and $\delta\rho$). But ε_2^a will become a sum of a U –ic and a ρ –ic piece. If so, the latter must be a linear combination of the ε_1^a ’s. At this point, new variables h and u are in order,

$$\begin{aligned} \varepsilon_1^a &= dh^a \quad , \quad \varepsilon_2^a = Q^{ab} dh^b + du^a \quad : \\ \left[\prod_a \varepsilon_1^a \varepsilon_2^a \right] &= \left[\prod_a dh^a du^a \right] \left| \det \begin{pmatrix} 1 & 0 \\ Q^{ab} & 1 \end{pmatrix} \right| = \left[\prod_a dh^a \right] \left[\prod_a du^a \right] \quad , \end{aligned} \quad (8.2)$$

to reach the desired decomposition.

Integration over du will give the gauge orbit \mathcal{G}_* , and the remaining measure $d\mu(\mathcal{H})$ might be related to the dh ’s. Other than in [2.13], [2.14] there appear no wedge products or \sim signs in the present derivation. But, admittedly, it is high time to see a figure.



For detailing the above, we have from (7.9), (7.6) and (8.1) that

$$\begin{aligned} \varepsilon^a &= 4 \operatorname{Tr} (T^a [\delta U \rho + U \delta \rho] \rho^{-1} U^\dagger) = 4 \operatorname{Tr} (\bar{T}^a U^\dagger \delta U) + 4 \operatorname{Tr} (\bar{T}^a \delta \rho \rho^{-1}) \\ \varepsilon^{a*} &= -4 \operatorname{Tr} (\bar{T}^a U^\dagger \delta U) + 4 \operatorname{Tr} (\bar{T}^a \rho^{-1} \delta \rho) \quad , \quad \bar{T}^a := U^\dagger T^a U \quad , \end{aligned} \quad (8.3)$$

where $[\operatorname{Tr}(A)]^* = \operatorname{Tr}(A^\dagger)$ and $\delta U^\dagger U = -U^\dagger \delta U$ have been used. One can see already that,

in the real part, U -ic terms will compensate:

$$\begin{aligned}\varepsilon_1^a &= \frac{\varepsilon^a + \varepsilon^{a*}}{2} = 2 \operatorname{Tr} \left(\overline{T}^a \rho^{-1} [\rho \delta \rho + \delta \rho \rho] \rho^{-1} \right) \\ &= 2 \operatorname{Tr} \left(\overline{T}^a H^{-1/2} \delta H H^{-1/2} \right) =: dh^a \quad .\end{aligned}\tag{8.4}$$

But there remain two terms in the imaginary part:

$$\begin{aligned}\varepsilon_2^a &= \frac{\varepsilon^a - \varepsilon^{a*}}{2i} = -2i \operatorname{Tr} \left(\overline{T}^a [\delta \rho \rho^{-1} - \rho^{-1} \delta \rho] \right) + du^a \\ \text{with } du^a &:= -4i \operatorname{Tr} \left(\overline{T}^a U^\dagger \delta U \right) = -4i \operatorname{Tr} \left(T^a \delta U U^\dagger \right) \quad .\end{aligned}\tag{8.5}$$

Below (5.8) it was understood that ρ is determined by n real parameters. Hence, every single-infinitesimal pure ρ -ic quantity — such as the corresponding term in (8.5) — can be linearly combined from dh^a 's:

$$-2i \operatorname{Tr} \left(\overline{T}^a [\delta \rho \rho^{-1} - \rho^{-1} \delta \rho] \right) =: Q^{ab} dh^b \quad .\tag{8.6}$$

We have thus obtained that $\varepsilon_2^a = Q^{ab} dh^b + du^a$, hence (8.2) is valid. One could object that there is still some dependence on U in (8.8), hidden in the generators $\overline{T}^a := U^\dagger T^a U$. Yes. But the whole matrix Q drops out in (8.2)!

It remains to relate the physical part $\prod_a dh^a$ of the volume with the measure $d\mu(\mathcal{H})$ in the space of hermitean unit-determinant matrices H . In order that the neighbour matrix $H + \delta H$ is hermitean too, we must write

$$H + \delta H = H^{1/2} (1 + \eta^a \overline{T}^a) H^{1/2} \quad , \quad \eta^a \text{ reell} \quad .\tag{8.7}$$

Linear in η^a we even have $\det(H + \delta H) = 1$ due to $\operatorname{Tr}(\overline{T}^a) = 0$. Solving (8.7) for η and comparing with (8.4),

$$\eta^a = 2 \operatorname{Tr} \left(\overline{T}^a H^{-1/2} \delta H H^{-1/2} \right) = dh^a \quad ,\tag{8.8}$$

exhibits the simplest possibility to be true. The measure in the space \mathcal{H} is thus given by

$$d\mu(\mathcal{H}) = \prod_{\vec{r}} \prod_a dh^a \quad , \quad \text{and} \quad d\mu(\mathcal{M}) = \prod_{\vec{r}} \prod_a du^a \cdot d\mu(\mathcal{H})\tag{8.9}$$

states the connection to $d\mu(\mathcal{M})$. In passing, the metrics in the \mathcal{H} space can be written as

$$ds_{\mathcal{H}}^2 = \int d^2 r \eta^a \eta^a = \int d^2 r 2 \operatorname{Tr} \left(\eta^a \overline{T}^a \overline{T}^b \eta^b \right) = 2 \int d^2 r \operatorname{Tr} \left(H^{-1} \delta H H^{-1} \delta H \right) \quad .\tag{8.10}$$

(8.10) agrees with [2.17]. But KKN state something else in place of (8.8), namely $2 \operatorname{Tr}(T^a H^{-1} \delta H)$, and call it the *Haar measure*. Wether there is a real difference or not (!) we come back to in § 14.3.

With view to (8.8) one may again object that there is reminescent U dependence hidden in \overline{T}^a . This time it is removed by observing that $\eta_{\text{non}}^a := 2 \operatorname{Tr}(T^a H^{-1/2} \delta H H^{-1/2})$

and η^a are related by an ordinary real rotation matrix. To realize this, we define \mathcal{D} by $\eta^a =: \mathcal{D}^{ab} \eta_{\text{non}}^b$ and read off from (8.10) that

$$\eta_{\text{non}}^a \eta_{\text{non}}^a = \eta^a \eta^a = \eta_{\text{non}}^b (\mathcal{D}^T)^{ba} \mathcal{D}^{ac} \eta_{\text{non}}^c \quad \leadsto \quad \mathcal{D}^T \mathcal{D} = 1 \quad . \quad (8.11)$$

To make use of this, we start from (8.9), put integrals in front of the relation to the right, rotate in the integrals over $d\mu(\mathcal{H})$ until their memory on U has gone, and finally shift the integrals over du^a to the right :

$$\int \int d\mu(\mathcal{M}) = \int d\mu(\mathcal{H}) \int \prod_{\vec{T}} \prod_a du^a = \int d\mu(\mathcal{H}) \cdot \int d\mu(\mathcal{G}_*) \quad . \quad (8.12)$$

There it is, the gauge volume $\int d\mu(\mathcal{G}_*)$, to be removed before quantizing.

Let us summarize the essence of the last three sections and combine the equations (6.1), (7.16) and (8.12) :

$$d\mu(\mathcal{C}) = \frac{d\mu(\mathcal{A})}{d\mu(\mathcal{G}_*)} = \frac{d\mu(\mathcal{M}) \det(\mathbf{D}^\dagger \mathbf{D})}{d\mu(\mathcal{G}_*)} = d\mu(\mathcal{H}) \det(\mathbf{D}^\dagger \mathbf{D}) \quad . \quad (8.13)$$

(8.13) is [2.19] : *the problem is thus reduced to the calculation of the determinant of the two-dimensional operator $\mathbf{D}^\dagger \mathbf{D}$.*

Perhaps, while recapitulating the present section, one could be dissatisfied with the argument leading to (8.6) which merely states the existence of the matrix Q . But Q can be made a bit more explicit also, e.g. by using the ρ representation (5.8) :

$$\begin{aligned} \rho &= e^{\vec{\omega} \vec{T}} \quad , \quad \delta \rho = d\omega^a \partial_{\omega^a} e^{\vec{\omega} \vec{T}} = d\omega^a \int_0^1 ds e^{s \vec{\omega} \vec{T}} [\partial_{\omega^a} \vec{\omega} \vec{T}] e^{(1-s) \vec{\omega} \vec{T}} \\ \delta \rho \rho^{-1} &= d\omega^a \int_0^1 ds \tau^a(s) \quad , \quad \tau^a(s) := e^{s \vec{\omega} \vec{T}} T^a e^{-s \vec{\omega} \vec{T}} \\ \rho^{-1} \delta \rho &= [\delta \rho \rho^{-1}]^\dagger = d\omega^a \int_0^1 ds \tau^{a\dagger}(s) \quad . \end{aligned} \quad (8.14)$$

With these details at hand (\vec{T}^a is now called T^a), (8.4) and (8.6) turn into

$$\begin{aligned} dh^a &= 2 S^{ab} d\omega^b \\ Q^{ab} dh^b &= -2 R^{ab} d\omega^b \end{aligned} \quad \text{with} \quad \left. \begin{matrix} S^{ab} \\ R^{ab} \end{matrix} \right\} := \int_0^1 ds \text{Tr} \left(T^a \left\{ \begin{matrix} [\tau^b + \tau^{b\dagger}] \\ i [\tau^b - \tau^{b\dagger}] \end{matrix} \right\} \right) \quad . \quad (8.15)$$

Eliminating $d\omega^b$ we have :

$$2 d\vec{\omega} = S^{-1} d\vec{h} \quad : \quad Q = -R S^{-1} \quad . \quad (8.16)$$

A quantity, which can be booked down, does exist.

9 Functional differential equations for $S[H]$

The material of this long section strongly defended itself to be understood. Sometimes, our way out is probably not the best.

Following the KKN guide [1] the somewhat delicate relation [2.20] is in order,

$$e^\Gamma = \det(\mathbf{D}^\dagger \mathbf{D}) = \sigma^n e^{2NS} \quad \text{with} \quad \sigma = \frac{\det'(-\bar{\partial}\partial)}{\int d^2r} \quad , \quad (9.1)$$

which presumably needs a regularization philosophy before any closer look at (see § 9.4 and § 11). For the stage being², let the factor σ contain all that is left for vanishing fields A . Then, all non-trivial A dependence is in the exponential, i.e. in S , and we have the condition

$$\lim_{A \rightarrow 0} S = \lim_{H \rightarrow \text{const}} S[H] = 0 \quad . \quad (9.2)$$

For the details of this line note that, as $\det(\mathbf{D}^\dagger \mathbf{D})$ might be a gauge invariant, S should depend on the physical degrees of freedom H only. This will be seen more explitley (see the headlines) and, admittedly, had been used already in splitting off the gauge volume. Due to $A = -(\partial M)M^{-1}$ a vanishing A corresponds to a constant matrix M , and this in turn (due $H = M^\dagger M$) to a constant H . (9.2) may be viewed to give initial values to the functional first order differential equation to be derived. The reader may look forward to (10.1) below to see the condition respected.

“Differentiate, regularize and integrate up again” is the general rule for obtaining S . It seems to be familiar to those experienced in anomaly calculations [11]. Under variation with respect to A fields,

$$\Gamma = \ln [\det(\mathbf{D}^\dagger \mathbf{D})] = n \ln(\sigma) + 2NS \quad \curvearrowright \quad \delta \Gamma = 2N \delta S \quad , \quad (9.3)$$

the stressy factor σ in (9.1) becomes irrelevant.

9.1 $\delta \ln(\det)$

For calculating $\delta \Gamma$, a true pedestrian recalls (7.13) and goes ahead³:

$$\Gamma = \ln [\det(\mathbf{D}^\dagger \mathbf{D})] = \hat{\mathbf{Tr}} [\ln(\mathbf{D}^\dagger \mathbf{D})] \quad , \quad 1 - \mathbf{D}^\dagger \mathbf{D} =: \mathcal{U}$$

² So far (April 18, 2000), the value of σ given to the right in (9.1) is not understood. May be that it leads into higher spheres [9, 10]. The determinant of a positive definit operator is, of course, the product of its eigenvalues: $\det(P) = \exp(\text{Tr}[\ln(P)]) = \exp(\sum \ln(\lambda)) = \prod \lambda$. Especially, $-\bar{\partial}\partial = -\Delta/4$ has $\lambda = \vec{k}^2/4$, and $\vec{k} = 0$ might be excluded. Also, \det' needs a large- k regularization. The power n is clear, too, – but that area in the denominator of σ remains a mystery.

³ Through $\delta \text{Tr}(\ln(X)) = \text{Tr}(\delta X/X)$, the second line of (9.5) can be reached immediately, of course. Bute we like to get used to the giant matrices.

$$= \hat{\mathbf{Tr}} \left(- \sum_{n=1}^{\infty} \frac{1}{n} \mathcal{U}^n \right) = - \sum_{n=1}^{\infty} \frac{1}{n} \begin{pmatrix} \mathcal{U}_{n1} & \mathcal{U}_{12} & \mathcal{U}_{23} \\ \mathcal{U}_{n-1\ n} & \overset{n}{\text{factors}} & \mathcal{U}_{34} \\ \bullet & \bullet & \bullet \end{pmatrix} . \quad (9.4)$$

Let a boldface \mathbf{Tr} refer to rr' indexed giant matrices, and the hat to their ab indices. E.g. the index 1 in (9.4) represents b, \vec{r}' , 2 stands for c, \vec{r}'' and so forth. A variation δ grasps into each of the n buckets \mathcal{U} :

$$\begin{aligned} \delta \Gamma &= - \sum_{n=1}^{\infty} [\delta \mathcal{U}_{12}] \mathcal{U}_{23} \dots \mathcal{U}_{n1} = - [\delta \mathcal{U}_{12}] \left(\frac{1}{1 - \mathcal{U}} \right)_{21} \\ &= \hat{\mathbf{Tr}} ([\delta(\mathbf{D}^\dagger \mathbf{D})] \mathbf{D}^{-1} \mathbf{D}^{\dagger-1}) = \hat{\mathbf{Tr}} (\mathbf{D}^{-1} \mathbf{D}^{\dagger-1} \delta(\mathbf{D}^\dagger \mathbf{D})) . \end{aligned} \quad (9.5)$$

If only one \mathbf{D} changes under the variation δ (\mathbf{D} itself, say), then the other (which is \mathbf{D}^\dagger) drops out in (9.5). Hence, when differentiating with respect to $A^a(\vec{r})$, the result is

$$\begin{aligned} \delta_{A^a(\vec{r})} \Gamma &= \hat{\mathbf{Tr}} (\mathbf{D}^{-1} \delta \mathbf{D}) = (D^{-1})_{12} (\delta D)_{21} = \int' \int'' (D^{-1})_{r'r''}^{bc} \delta_{A^a(\vec{r})} D_{r''r'}^{cb} \\ &= \int' \int'' (D^{-1})_{r'r''}^{bc} \delta(\vec{r} - \vec{r}'') f^{abc} \delta(\vec{r}'' - \vec{r}') \\ \delta_{A^a(\vec{r})} \Gamma &= f^{abc} (D^{-1})_{r'r}^{bc} \Big|_{\vec{r}' \rightarrow \vec{r}} , \quad \delta_{A^{a*}(\vec{r})} \Gamma = f^{abc} (D^{*-1})_{r'r}^{bc} \Big|_{\vec{r}' \rightarrow \vec{r}} . \end{aligned} \quad (9.6)$$

In the second line, there are two delta functions. The first arised by functional differentiating the A field in $D_{r''r'}^{cb} = \delta^{cb} \partial_{r''r'} + A^\bullet(\vec{r}'') f^{c\bullet b} \delta_{r''r'}$. It is harmless: one may just integrate over \vec{r}'' . The second delta is the one which was already present in $D_{r''r'}^{cb}$. If integrating, it would make the two spatial indices on D^{-1} equal. In the third line, following KKN, we postpone this dangerous coincidence. It must wait for the regularizaition of § 11. In the last line, since Γ is real, the second differential equation was added as the c.c. of the first.

9.2 The inverse of \mathbf{D}

The inverse of \mathbf{D} , required in (9.6) and now denoted by $(?)$, is the unique solution of the n^2 equations

$$D_{rr''}^{ac} (?)_{r'r'}^{cb} = \left(\delta^{ac} \partial + A^{ac}(\vec{r}) \right) (?)_{rr'}^{cb} = \delta^{ab} \delta(\vec{r} - \vec{r}') \quad \text{with } A^{ac} := A^\bullet f^{a\bullet c} . \quad (9.7)$$

We like inventing. Due to (4.3), i.e. $\partial G_{rr'} = \delta(\vec{r} - \vec{r}')$, the quantity $(?)$ might be anyhow related to G . If we insert $\delta^{cb} G_{rr'}$ for $(?)$, a $\delta\delta$ arises to the right but also an additional term AG . To compensate the latter, we make the next attempt with $(?) = M^{cb}(\vec{r}) G_{rr'}$. This time the r.h.s. becomes $(\partial M^{ab}) G + M^{ab} \delta_{rr'} + A^{ac} M^{cb} G$. No. But we see that a further matrix $(M^{-1})^{db}(\vec{r}')$ would restore the $\delta\delta$:

$$\left(\delta^{ac} \partial + A^{ac}(\vec{r}) \right) M_r^{cd} G_{rr'} (M^{-1})_{r'}^{db} = \left[(\partial M)_r^{ad} + A_r^{ac} M_r^{cd} \right] G_{rr'} (M^{-1})_{r'}^{db} + \delta^{ab} \delta_{rr'} . \quad (9.8)$$

The so far arbitrary matrix M^{ab} can now be fixed to make the square bracket vanish. If we are able to construct the “adjoint” matrix \hat{M} having elements M^{ab} with the property announced, i.e. if the equation $\hat{A}\hat{M} = -\partial\hat{M}$ can be solved, then the solution to (9.7) reads :

$$(D^{-1})_{rr'}^{ab} = M_r^{ac} G_{rr'}(M^{-1})_{r'}^{cb} \quad , \quad \text{or} \quad \mathbf{D}^{-1} = \hat{M} G \hat{M}^{-1} \quad . \quad (9.9)$$

The elements of the matrix \hat{A} are those given to the right in (9.7). The equation $\hat{A}\hat{M} = -\partial\hat{M}$ to be solved may be called the adjoint version of the familar relation $AM = -\partial M$. By the way, do not integrate over pairs of spatial indices in (9.8) and (9.9) (the last pair to be integrated occurred to the very left in (9.7)).

9.3 The adjoint matrix \hat{M}

We start from the fundamental representation, i.e. from $A = -iT^a A^a = -(\partial M)M^{-1}$, and put this in the commutator $[T^b, \dots]$ to obtain

$$A^{bc}T^c = -T^b(\partial M)M^{-1} + (\partial M)M^{-1}T^b = -T^b(\partial M)M^{-1} - M(\partial M^{-1})T^b \quad . \quad (9.10)$$

Multiplication by M^{-1} from the left and by M from the right gives

$$\begin{aligned} A^{bc}M^{-1}T^cM &= -M^{-1}T^b\partial M - (\partial M^{-1})T^bM = -\partial M^{-1}T^bM \quad , \\ \text{i.e. } A^{bc}\text{Tr}(M^{-1}T^cMT^d) &= -\partial\text{Tr}(M^{-1}T^bMT^d) \quad , \end{aligned} \quad (9.11)$$

which allows for reading off the solution to $\hat{A}\hat{M} = -\partial\hat{M}$ only up to normalization. For the latter we require $M^{ab} \rightarrow \delta^{ab}$ for $A \rightarrow 0$ in order to make (4.4) or (4.5) valid even adjointly. To summarize :

$$\hat{M} : M^{ab} = 2\text{Tr}(T^aMT^bM^{-1}) \quad \text{solves} \quad \hat{A}\hat{M} = -\partial\hat{M} \quad . \quad (9.12)$$

Because of $(\hat{M}^\dagger)^{ab} = M^{ba*} = 2\text{Tr}(M^{\dagger-1}T^aM^\dagger T^b)$ one may add that

$$\hat{M}^\dagger : (M^\dagger)^{ab} = 2\text{Tr}(T^aM^\dagger T^bM^{\dagger-1}) \quad \text{solves} \quad \hat{M}^\dagger\hat{A}^\dagger = -\bar{\partial}\hat{M}^\dagger \quad . \quad (9.13)$$

As \hat{A} has the elements $A^{ab} = A^{\bullet*}f^{a\bullet b}$, \hat{A}^\dagger has the elements $(A^\dagger)^{ab} = A^{\bullet*}f^{b\bullet a} = -A^{\bullet*}f^{a\bullet b}$.

9.4 Jump into the regularized version

Disgraceful ! But guided by KKN, and because things are worked out in § 11, let us be allowed to anticipate the result of regularization here. Remember from (9.6) and (9.9) that it is the Greens function $G_{rr'}$ suffering under the coincidence limit $\vec{r}' \rightarrow \vec{r}$. G has to be replaced by its regularized version $\mathcal{G}_{rr'}$. Then the coincidence limit can be performed, see (11.28), to give

$$\left[D_{r'r}^{-1} \right]_{\vec{r}' \rightarrow \vec{r}}^{\text{reg}} = -\frac{\hat{A}^\dagger - (\bar{\partial}\hat{M})\hat{M}^{-1}}{\pi} \quad , \quad \left[D_{r'r}^{*-1} \right]_{\vec{r}' \rightarrow \vec{r}}^{\text{reg}} = \frac{\hat{A} - \hat{M}^{\dagger-1}\partial\hat{M}^\dagger}{\pi} \quad . \quad (9.14)$$

Herewith the ill equations (9.6) turn into well behaved, local (but still adjointly formulated) functional differential equations, namely :

$$\delta^a \Gamma = \frac{-1}{\pi} f^{abc} \left(\hat{A}^\dagger - (\bar{\partial} \hat{M}) \hat{M}^{-1} \right)^{bc}, \quad \delta^{a*} \Gamma = \frac{1}{\pi} f^{abc} \left(\hat{A} - \hat{M}^{\dagger-1} \partial \hat{M}^\dagger \right)^{bc}. \quad (9.15)$$

Here, of course, δ^a stands for $\delta_{A^a(\vec{r})}$ and δ^{a*} for $\delta_{A^{a*}(\vec{r})}$. The right equation is the c.c. of the left one. Our next task is the return to the more familiar $N \times N$ matrices M . Adjoint traces might become fundamental ones.

9.5 Trace times trace – back to fundamental

There is a nice technical detail on traces, which can be discovered by searching the inverse of \hat{M} . It has to be determined from $(M^{-1})^{ac} M^{cb} = \delta^{ab}$. We guess the result,

$$(M^{-1})^{ac} = 2 \operatorname{Tr} (T^a M^{-1} T^c M) = M^{ca}, \quad (9.16)$$

and start checking. Two fundamental traces are to be combined :

$$\begin{aligned} (M^{-1})^{ac} M^{cb} &= 2 \operatorname{Tr} (T^a M^{-1} T^c M) - 2 \operatorname{Tr} (T^c M T^b M^{-1}) \\ &= 2 \operatorname{Tr} ([MT^a M^{-1}] T^c) - 2 \operatorname{Tr} (T^c \{MT^b M^{-1}\}) \\ &= 2 [MT^a M^{-1}]_{\ell m} \underbrace{2 T_{m\ell}^c T_{pq}^c}_{\substack{= \delta_{mq} \delta_{\ell p} \\ - \frac{1}{N} \delta_{m\ell} \delta_{pq}}} \{MT^b M^{-1}\}_{qp} \\ &= 2 \operatorname{Tr} ([MT^a M^{-1}] \{MT^b M^{-1}\}) = \delta^{ab} \quad \text{q.e.d.} \end{aligned} \quad (9.17)$$

The $1/N$ term dropped out since $\operatorname{Tr}(MT^a M^{-1}) = \operatorname{Tr}(MT^b M^{-1}) = 0$. The trick will be repeatedly used in the sequel. Having Miss Maple in mind, we call it *concatenation*. To summarize, $\operatorname{Tr}(AT^a) - 2 \operatorname{Tr}(T^a B) = \operatorname{Tr}(AB)$, if either $\operatorname{Tr}(A) = 0$ or $\operatorname{Tr}(B) = 0$ or both.

Already in the next step, namely fundamentalizing (9.15), *concatenation* is at work :

$$\begin{aligned} (\bar{\partial} M^{bd}) (M^{-1})^{dc} &= 2 \operatorname{Tr} (T^b (\bar{\partial} M) T^d M^{-1} + T^b M T^d \bar{\partial} M^{-1}) - 2 \operatorname{Tr} (T^d M^{-1} T^c M) \\ &= 2 \operatorname{Tr} (M^{-1} [T^b \bar{\partial} M - (\bar{\partial} M) M^{-1} T^b M] T^d) - 2 \operatorname{Tr} (T^d M^{-1} T^c M) \\ &= 2 \operatorname{Tr} ([T^b (\bar{\partial} M) M^{-1} - (\bar{\partial} M) M^{-1} T^b] T^c) \\ &= 2 \operatorname{Tr} ([T^c, T^b] (\bar{\partial} M) M^{-1}) = -2i f^{abc} \operatorname{Tr} (T^a (\bar{\partial} M) M^{-1}), \\ (M^{\dagger-1})^{bd} \partial (\hat{M}^\dagger)^{dc} &= \dots = -2i f^{abc} \operatorname{Tr} (T^a M^{\dagger-1} \partial M^\dagger). \end{aligned} \quad (9.18)$$

But it is rather trivial to fundamentalize the A terms in (9.15) :

$$A^{bc} = A^a f^{bac} = -2i f^{abc} \operatorname{Tr} (T^a A), \quad (A^\dagger)^{bc} = -A^{\bullet*} f^{b\bullet c} = -2i f^{abc} \operatorname{Tr} (T^a A^\dagger). \quad (9.19)$$

The f factors in (9.18), (9.19) are highly welcome, because, since (9.3), $\delta \Gamma =: 2NS$, we mused on the factor N . Now it turns about through $f^{abc} f^{bc\bullet} = N \delta^{a\bullet}$. Using (9.19) and (9.18) in (9.15) one obtains

$$\delta^a \Gamma = 2N \frac{i}{\pi} \operatorname{Tr} (T^a [A^\dagger - (\bar{\partial} M) M^{-1}]), \quad \delta^{a*} \Gamma = -2N \frac{i}{\pi} \operatorname{Tr} (T^a [A - M^{\dagger-1} \partial M^\dagger]). \quad (9.20)$$

(9.20), right equation, is [2.23] and the c.c. of the left. Of course, the functional differential equations for S are given by (9.20) by simply omitting the prefactors $2N$. But they have by far not yet the appropriate form.

9.6 $S = S_1 + S_2$: the differential equations for S_2

Reformulation, one more reformulation — to which end? Let us have a look ahead to the first equation of § 10. The solution should be a functional of H only, $S[H]$, and anyhow this might be seen already in the differential equations. Things have been as in a chess game : three good moves, each one followed by three further good moves, and so on. Maybe, the actual way gone (with some temporal inflation of notations, sorry for) reflects a bit of these difficulties.

In (9.20) there occur quantities with “good” differentiation (∂ likes M , and $A = -(\partial M)M^{-1}$ is good) and with a “false” one (therefore F) :

$$F := -(\bar{\partial}M)M^{-1} \quad , \quad F^\dagger = -M^{\dagger-1}\partial M^\dagger \quad . \quad (9.21)$$

To repeat (9.20) it is very convenient to write the generators as integrated delta functions $T^a = i \int \delta^a A$ and $T^{a*} = -i \int \delta^{a*} A^\dagger$. Then

$$\delta^a S = -\frac{1}{\pi} \int \text{Tr} \left([A^\dagger + F] \delta^a A \right) \quad , \quad \delta^{a*} S = -\frac{1}{\pi} \int \text{Tr} \left([A + F^\dagger] \delta^{a*} A^\dagger \right) \quad . \quad (9.22)$$

This shows $2 * n$ functional differential equations for S . \int stands for $\int d^2 r$ over the whole 2D plane. \vec{r} is integration variable. Hence (to distinguish variables) we read δ^a as $\delta_{A^a(\vec{r}_0)}$. In (9.22) we may replace $\delta^a A$ by $\delta^a [A + F^\dagger]$, and $\delta^{a*} A^\dagger$ by $\delta^{a*} [A^\dagger + F]$, because

$$A, F \quad \text{depend on only} \quad A^a \quad \text{and} \quad A^\dagger, F^\dagger \quad \text{on only} \quad A^{a*} \quad .$$

To realize this, remember (4.5) showing the expansion of M (hence $\bar{\partial}M$ as well) in powers of A^a . The next abbreviation, which is $\Omega := A + F^\dagger$, shortens (9.22) to

$$\delta^a(2\pi S) = \int \text{Tr} (-2 \Omega^\dagger \delta^a \Omega) \quad , \quad \delta^{a*}(2\pi S) = \int \text{Tr} (-2 \Omega \delta^{a*} \Omega^\dagger) \quad . \quad (9.23)$$

It is tempting here to begin with guessing. But the attempt with $\int \text{Tr}(\Omega \Omega^\dagger)$ only leads to a reasonable decomposition :

$$2\pi S = 2\pi S_1 + 2\pi S_2 \quad , \quad 2\pi S_1 := \int \text{Tr} (-\Omega \Omega^\dagger) \quad . \quad (9.24)$$

As will be seen, S_1 remains the harmless first term of the action S , while S_2 advances to become the volume term of the WZW action. Conveniently we study the functional differential equations for the part $S_2 = S - S_1$ separately. Hence $\delta(2\pi S_1) = \int \text{Tr} (-\Omega^\dagger \delta \Omega - \Omega \delta \Omega^\dagger)$ has to be subtracted from (9.23) to give

$$\delta^a(2\pi S_2) = \int \text{Tr} (\Omega \delta^a \Omega^\dagger - \Omega^\dagger \delta^a \Omega) \quad , \quad \delta^{a*}(2\pi S_2) = \int \text{Tr} (\Omega^\dagger \delta^{a*} \Omega - \Omega \delta^{a*} \Omega^\dagger) \quad . \quad (9.25)$$

9.7 Variables H only

If S depends on only $H = M^\dagger M$ then we expect the differential equations to depend on only H , $\delta^a H$ and $\delta^{a*} H$:

$$\begin{aligned}\Omega &= A + F^\dagger = - \left[(\partial M) M^{-1} + M^{\dagger-1} (\partial M^\dagger) \right] \\ &= -M^{\dagger-1} \left[M^\dagger \partial M + (\partial M^\dagger) M \right] M^{-1} = -M^{\dagger-1} (\partial H) M^{-1} , \\ \Omega^\dagger &= -M^{\dagger-1} (\bar{\partial} H) M^{-1} = M (\bar{\partial} H^{-1}) M^\dagger .\end{aligned}\tag{9.26}$$

Herewith, the part S_1 in (9.24) becomes a H -ic object immediately :

$$2\pi S_1 = \int \text{Tr} \left((\partial H) \bar{\partial} H^{-1} \right) .\tag{9.27}$$

But for S_2 we must keep track with the differential equations, e.g. with the left equation (9.25). It becomes H -ic through

$$\begin{aligned}\delta^a (2\pi S_2) &= \int \text{Tr} \left(M^{\dagger-1} (\partial H) M^{-1} \delta^a \left[M^{\dagger-1} (\bar{\partial} H) M^{-1} \right] - \text{ditto}_{\partial \leftrightarrow \bar{\partial}} \right) \\ &= \int \text{Tr} \left((\partial H) H^{-1} \delta^a \left[(\bar{\partial} H) H^{-1} \right] - \text{ditto}_{\partial \leftrightarrow \bar{\partial}} \right) \\ &= \frac{i}{2} \int \text{Tr} \left(H_2 H^{-1} \delta^a \left[H_1 H^{-1} \right] - \text{ditto}_{1 \leftrightarrow 2} \right) \\ &= \frac{i}{2} \int \text{Tr} \left(H^{-1} H_2 H^{-1} \delta^a H_1 - H^{-1} H_2 H^{-1} H_1 H^{-1} \delta^a H - \text{ditto}_{1 \leftrightarrow 2} \right) \\ &= \frac{i}{2} \int \text{Tr} \left(\varphi^a \left[X_1 X_2 - X_2 X_1 \right] + \partial_1 (X_2 \varphi^a) - \partial_2 (X_1 \varphi^a) \right) ,\end{aligned}\tag{9.28}$$

where the abbreviations in the last line are

$$X_j := H^{-1} H_{ij} \quad (j = 1, 2) \quad \text{and} \quad \varphi^a := H^{-1} \delta^a H .\tag{9.29}$$

In the first line of (9.28) we were allowed to commute $M^{\dagger-1}$ with δ^a . The third line was obtained through “ $\partial \bar{\partial} - \bar{\partial} \partial = \frac{1}{4}(\partial_1 + i\partial_2)(\partial_1 - i\partial_2) - \frac{1}{4}(\partial_1 - i\partial_2)(\partial_1 + i\partial_2) = \frac{i}{2} [\partial_2 \partial_1 - \partial_1 \partial_2]$ ”. In the last line of (9.28), finally,

$$\begin{aligned}X_2 H^{-1} \delta^a H_1 &= X_2 H^{-1} \partial_1 \delta^a H = \partial_1 (X_2 \varphi^a) - (H^{-1} H_2 H^{-1})_1 \delta^a H \\ &= \partial_1 (X_2 \varphi^a) - H^{-1} H_{21} \varphi^a + [X_1 X_2 + X_2 X_1] \varphi^a ,\end{aligned}$$

was used, whereof 1-2-symmetric terms drop out. Clearly, the desired H -ic version is reached.

9.8 Stokes — a common δ

In the result (9.28), last line, one recognizes the third component of a curl operator. We now add the equation for $\delta^{a*} S$, which follows from an analogous calculation or simply

as the conjugate complex of $\delta^a S_2$:

$$\begin{aligned}\delta^a S_2 &= \frac{i}{4\pi} \int \text{Tr} \left(\varphi^a [X_1 X_2 - X_2 X_1] + \left[\nabla \times \left(X_1 \varphi^a, X_2 \varphi^a, \dots \right) \right]_3 \right) \\ \delta^{a*} S_2 &= \frac{i}{4\pi} \int \text{Tr} \left(\psi^a [X_1 X_2 - X_2 X_1] - \left[\nabla \times \left(X_1 \psi^a, X_2 \psi^a, \dots \right) \right]_3 \right),\end{aligned}\quad (9.30)$$

where $\psi^a := H^{-1} \delta^{a*} H$, which is just φ^a with δ^{a*} in place of δ^a . Is there any other small difference? For the first term together with the l.h.s. of each line we may in fact introduce a common δ , which may be freely chosen to be $\delta_{A^a(\vec{r}_0)}$ or $\delta_{A^{a*}(\vec{r}_0)}$. In place of φ^a , ψ^a a comon $\varphi := H^{-1} \delta H$ would be sufficient. But in the curl term the difference in sign prevents from such unification.

$\nabla \times$ calls for Stokes theorem (for a plane, in the case (9.30) at hand). If there is nothing left at the border of the infinite 2D plane, the curl terms in (9.30) may be omitted, and we arrive at

$$\begin{aligned}\delta S_2 &= \frac{i}{4\pi} \int \text{Tr} \left(\varphi [X_1 X_2 - X_2 X_1] \right), \quad \varphi = H^{-1} \delta H \\ &= \frac{i}{4\pi} \int \text{Tr} \left(H^{-1} \delta H \left[H^{-1} H_{r_1} H^{-1} H_{r_2} - H^{-1} H_{r_2} H^{-1} H_{r_1} \right] \right),\end{aligned}\quad (9.31)$$

which is the final result of this lengthy § 9.

For safety, however, let us look at possible border terms, Stokes could have left. Note that $\varphi^a = H^{-1} \delta^a H$ is a function of two spatial variables. One is \vec{r} , H depends on and $\nabla \times$ acts upon. The other is \vec{r}_0 , the position mark in $\delta^a = \delta_{A^a(\vec{r}_0)}$. In addition, as H is also a functional of the fields A , there is also the integration variable \vec{r}' , see (4.4), (4.5). Through δ^a a \vec{r}' is forced at the position \vec{r}_0 . If now \vec{r} runs far away from \vec{r}_0 , the Greens functions G in (4.5) might care for φ^a vanishing at infinity.

$$\text{----- } \vec{r} \text{ border } \text{----- } A(\vec{r}') \text{ --- } \delta_{A^a(\vec{r}_0)} \text{----- } \text{border } \vec{r} \text{-----}$$

If, in addition, $A \rightarrow 0$ at $r \rightarrow \infty$ (H and $M \rightarrow \text{const}$), then even the factors X_j support the omission of border contributions due Stokes.

For later use, also the H -ic functional differential equations for S_1 and for $S = S_1 + S_2$ should be noticed. If starting from (9.23) and using (9.26) we obtain

$$\delta S_1 = \frac{1}{4\pi} \int \left\{ \text{Tr} \left(\varphi [\partial_1 X_1 + \partial_2 X_2] \right) + \partial_1 \text{Tr}(\varphi X_1) + \partial_2 \text{Tr}(\varphi X_2) \right\}, \quad (9.32)$$

i.e. a common δ immediately. This time it is Gauss, who removes the derivative terms in (9.32). To summarize, we have

$$\begin{aligned}\delta S_1 &= \frac{1}{2\pi} \int \text{Tr} \left(\varphi [\partial \bar{X} + \bar{\partial} X] \right), \quad X := H^{-1} \partial H, \quad \bar{X} := H^{-1} \bar{\partial} H \\ \delta S_2 &= \frac{1}{2\pi} \int \text{Tr} \left(\varphi [\bar{X} X - X \bar{X}] \right)\end{aligned}$$

$$\delta S = \frac{1}{2\pi} \int \text{Tr} \left(\varphi [\partial \bar{X} + \bar{\partial} X + \bar{X} X - X \bar{X}] \right) = \frac{1}{\pi} \int \text{Tr} \left(\varphi \partial \bar{X} \right), \quad (9.33)$$

because of the identity

$$\bar{\partial}X + \overline{X}X = \bar{\partial} [H^{-1}\partial H] + \overline{X}X = H^{-1}\partial\bar{\partial}H = \partial\overline{X} + X\overline{X} \quad . \quad (9.34)$$

Assume that all the nice (?) above notations survive the § 12, but are forgotten in § 14 of part II. So, here is equation (9.33) in normal life :

$$\delta S = \frac{1}{\pi} \int \text{Tr} \left(H^{-1}(\delta H) \partial H^{-1} \bar{\partial} H \right) \quad . \quad (9.35)$$

10 Solution $S[H]$ to the differential equations

Well, we like inventing. But at this moment the view to the alleged result [**2.21**] is really irresistible. The solution S to the above functional differential equations is the hermitean WZW action

$$S = \frac{1}{2\pi} \int \text{Tr} \left((\partial H) \bar{\partial} H^{-1} \right) + \frac{i}{12\pi} \int_V \epsilon^{jkl} \text{Tr} \left(H^{-1}(\partial_j H) H^{-1}(\partial_k H) H^{-1}(\partial_l H) \right) \quad (10.1)$$

The first term is S_1 , and is well known from (9.27). Thus, the second term is S_2 , and it is purely H -ic, too. It has to solve the differential equation (9.31). Obviously, both terms respect the condition (9.2) separately, as they vanish for $H \rightarrow \text{const}$.

10.1 The volume term

S_2 is a volume integral. How this ?? H is defined on the xy plane. We live on this plane and can't any other. Obviously, someone mad has given our H 's an additional dependence on the variable z . In fact, in her TFT'98 proceedings article [12] Dimitra Karabali states that “*the integrand thus requires an extension of the matrix field H into the interior of V , but physical results do not depend on how this extension is done. Actually for the special case of hermitian matrices the second term can also be written as an integral over [2D] spatial coordinates only.*” — aah — ? — .

Here, there is a pitfall to run into with ease. The volume integral makes sense, one could think, only if it is transformed into a surface integral by means of the Gauss theorem (a scalar integrand can always be written as $\nabla \cdot \vec{C}$). Then, the values of S_2 might lie on this surface, and now one can proceed with functional differentiation. But this is only one possibility of going ahead with checking the S_2 differential equation (we still believe that one can go this way, anyhow and at least formally). The other way was learned from [13] : functionally differentiate **f i r s t**, i.e. the content of the volume, and use Gauss only afterwards. Here we do what we can, and go the easy way.

As mentioned, let the hermitean matrices H anyhow depend on z . Now, apart from X_1, X_2 , there is also a $X_3 = H^{-1}H_{/3}$. In the ε -tensor language of (10.1) there are six terms under the trace. But cyclic permutations reduce to two:

$$S_2 = \frac{i}{4\pi} \int_V \text{Tr} \left(X_3 [X_1 X_2 - X_2 X_1] \right) . \quad (10.2)$$

Remember the common δ , and let it act at a position \vec{r}_0 anywhere in the volume V . Recalling also $\varphi = H^{-1}\delta H$, we are going to check whether (10.2) solves the differential equation (9.31):

$$\begin{aligned} \delta S_2 &= \frac{i}{4\pi} \int_V \text{Tr} \left([X_1 X_2 - X_2 X_1] \delta X_3 + \text{cyclic} \right) \\ &\quad \delta X_3 = \delta(H^{-1}H_{/3}) = X_3 \varphi - \varphi X_3 + \partial_3 \varphi , \\ &\quad \left[[X_1, X_2], X_3 \right] + \text{cyclic} = 0 \quad (\text{Jacobi identity}) \\ &= \frac{i}{4\pi} \int_V \text{Tr} \left([X_1, X_2] \partial_3 \varphi + \text{cyclic} \right) \\ &= \frac{i}{4\pi} \int_V \text{Tr} \left(\partial_3 \varphi [X_1, X_2] + \text{cyclic} - \varphi \{ \partial_3 [X_1, X_2] + \text{cyclic} \} \right) \\ &\quad X_1 X_2 - X_2 X_1 = \partial_2 X_1 - \partial_1 X_2 \quad \hookrightarrow \quad \{ \quad \} = 0 \\ &= \frac{i}{4\pi} \int_V \nabla \cdot \left(\text{Tr}(\varphi [X_2, X_3]) , \text{Tr}(\varphi [X_3, X_1]) , \text{Tr}(\varphi [X_1, X_2]) \right) \\ &\quad \textbf{Gauss but now :} \\ &= \frac{i}{4\pi} \int_{\partial V} d\vec{f} \cdot \left(\text{Tr}(\varphi [X_2, X_3]) , \text{Tr}(\varphi [X_3, X_1]) , \text{Tr}(\varphi [X_1, X_2]) \right) \\ \delta S_2 &= \frac{i}{4\pi} \int \text{Tr} \left(\varphi [X_1 X_2 - X_2 X_1] \right) \quad \text{on top of the surface ,} \quad \text{q.e.d.} \quad (10.3) \end{aligned}$$

The differential equations are fulfilled. Hence, all is done what the headline announces. The question mark at the end of the above Karabali text refers to the claim that S_2 can be written as a plane integral, too. But we had not the strength to verify this detail.

10.2 Recapitulation

Five sections have gone just to learn how a wave function ψ of the Schrödinger wave functional quantum mechanics might be normalized. The variables, ψ depends on, vary on “field axes” ($2n$ axes, at the beginning). Already in (6.2) the scalar product was noticed. But the question marks there were overcome while changing from the measure $d\mu(\mathcal{A})$ to $d\mu(\mathcal{C})$ in § 8, thereby removing the unphysical gauge volume, nothing depends on, even not the remaining weight $\exp[2NS[H]]$. To make this weight explicit, we were led into all the trouble with the “giant” Jacobian $\det(\mathbf{D}^\dagger \mathbf{D})$ in § 9. After all, the scalar product now reads

$$\int \psi_1^* \psi_2 = \int d\mu(\mathcal{C}) \psi_1^*[H] \psi_2[H] = \sigma^n \int d\mu(\mathcal{H}) e^{2NS[H]} \psi_1^*[H] \psi_2[H] . \quad (10.4)$$

(10.4) ist [2.25]. KKN: *This formula shows that all matrix elements in (2+1)-dimensional $SU(N)$ gauge theory can be evaluated as correlators of the hermitean WZW-Modell.* Note that the number of field axes has reduced to n , because with (8.14) $H = \rho^2 = e^{2\vec{\omega}\vec{T}}$ carries the n real parameters ω^a . The normalization prescription is inherent in (10.4). But a general wave functional ψ can never be normalizable, as we know well from non-relativistic quantum mechanics. At minimum, we will have to learn on this selection. And than there is the equation of motion $i\hbar\dot{\psi} = \mathbf{H}\psi$. How might the Hamiltonian \mathbf{H} look like in Schrödinger wave field theory ?

10.3 Polyakov–Wiegmann identity

To work with the action $S[H]$, one has not necessarily to go into the details of (10.1). The simpler property is its variation (9.35). There are even other properties, as its conformal invariance (see § 12.2) or the following *remarkable identity* [14] concerning a product argument. Its three-lines reasoning in [14] is hardly understood, and the result given there is wrong⁴. By [3.2] the identity is given incorrect even in [1]. So, let us do it right :

$$S[AB] = S[A] + S[B] - \frac{1}{\pi} \int \text{Tr} \left([\partial B] B^{-1} A^{-1} \bar{\partial} A \right) . \quad (10.5)$$

To derive (10.5), we write $S = S_1 + S_2$ again and obtain

$$\begin{aligned} S_1[AB] &= \frac{1}{2\pi} \int \text{Tr} \left([\partial AB] \bar{\partial} B^{-1} A^{-1} \right) \\ &= \frac{1}{2\pi} \int \text{Tr} \left([(\partial A)B + A\partial B] [(\bar{\partial} B^{-1})A^{-1} + B^{-1}\bar{\partial} A^{-1}] \right) \\ &= S_1[A] + S_1[B] - \frac{1}{2\pi} \int \text{Tr} \left((\bar{\partial} B)B^{-1}A^{-1}\partial A + (\partial B)B^{-1}A^{-1}\bar{\partial} A \right) \\ &= S_1[A] + S_1[B] - \frac{1}{4\pi} \int \text{Tr} \left(b_1 a_1 + b_2 a_2 \right) , \quad \begin{array}{l} a_j := A^{-1} A_{\iota j} \\ b_j := B_{\iota j} B^{-1} \end{array} . \end{aligned} \quad (10.6)$$

Turning to S_2 , using (10.2) and again replacing H by AB , we have $X_j = B^{-1}A^{-1}(AB)_{\iota j} = B^{-1}(a_j + b_j)B$ to be inserted there :

$$\begin{aligned} S_2[AB] &= \frac{i}{4\pi} \int_V \text{Tr} \left(X_3 [X_1 X_2 - X_2 X_1] \right) \\ &= \frac{i}{4\pi} \int_V \text{Tr} \left[(a_3 + b_3)(a_1 + b_1)(a_2 + b_2) - \text{ditto}_{\text{anticyclic}} \right] \\ &= S_2[A] + S_2[B] + \frac{i}{4\pi} \int_V \text{Tr} \left[\text{mixing terms} \right] . \end{aligned} \quad (10.7)$$

Among the six mixing terms there are three with one b times a commutator of two a 's, and three with one a and a commutator of b 's. The three a -commutators, for instance,

⁴ It is equation (7) there. The relation is stated in terms of a functional W , which is nowhere defined. If (5) is meant (with or without the prefactor $1/(2\pi)$), then (7) is wrong. Some authors might repeat their half-time examinations.

are the components of $\vec{a} \times \vec{a}$. Altogether we have [mixing terms] = $\vec{b}(\vec{a} \times \vec{a}) + (\vec{b} \times \vec{b})\vec{a}$ under the trace. This is very welcome, because through

$$\begin{aligned} \nabla \times \vec{b} = \nabla \times [(\nabla B)B^{-1}] &= \vec{b} \times \vec{b} \quad , \quad \nabla \times \vec{a} = \nabla \times [A^{-1}\nabla A] = -\vec{a} \times \vec{a} \quad , \\ \left[\text{mixing terms} \right] &= (\nabla \times \vec{b})\vec{a} - \vec{b}(\nabla \times \vec{a}) = \nabla(\vec{b} \times \vec{a}) \quad , \end{aligned} \quad (10.8)$$

Gauss helps us to get rid of the unwanted last volume integral. On the surface on top of the volume V it yields $(\vec{b} \times \vec{a})_3 = b_1 a_2 - b_2 a_1$, and in total

$$S_2[AB] = S_2[A] + S_2[B] - \frac{1}{4\pi} \int \text{Tr} \left(i b_2 a_1 - i b_1 a_2 \right) \quad . \quad (10.9)$$

Adding (10.9) to (10.6) gives the announced result

$$S[AB] = S[A] + S[B] - \frac{1}{4\pi} \int \text{Tr} \left((b_1 + i b_2)(a_1 - i a_2) \right) \equiv (10.5) \quad , \quad \text{q.e.d.} \quad (10.10)$$

Reporting him of this success, Sergei Ketov smiled : *yes, a pedestrian needs Gauss, but the elegant derivation is that in [14]*.

For a nice check of the identity (10.5) one may use it to obtain the differential equation (9.35) again.

$$\begin{aligned} \delta S &= S[H + \delta H] - S[H] = S \left[H (1 + H^{-1} \delta H) \right] - S[H] \\ &= S \left[1 + H^{-1} \delta H \right] - \frac{1}{\pi} \int \text{Tr} \left((\partial H^{-1} \delta H)(1 - H^{-1} \delta H) H^{-1} \bar{\partial} H \right) \\ &= \frac{1}{\pi} \int \text{Tr} \left(H^{-1} \delta H \partial H^{-1} \bar{\partial} H \right) \equiv (9.35) \end{aligned} \quad (10.11)$$

using partial integration and neglecting terms $\sim (\delta H)^2$.

In the next section we even need $S[ABC]$. But this is merely one more exercise on (10.5) :

$$\begin{aligned} S[ABC] &= S[A] + S[B] + S[C] - \frac{1}{\pi} \int \text{Tr} \left(\right. \\ &\quad \left. (\partial C) C^{-1} B^{-1} A^{-1} (\bar{\partial} A) B + (\partial C) C^{-1} B^{-1} \bar{\partial} B + (\partial B) B^{-1} A^{-1} \bar{\partial} A \right) \quad . \end{aligned} \quad (10.12)$$

While checking (10.12) with e.g. $ABC = HH^{-1}H$, one becomes aware of two more strange relations, namely

$$S_1[H^{-1}] = S_1[H] \quad , \quad S_2[H^{-1}] = -S_2[H] \quad , \quad (10.13)$$

and obtains $3S_1 + S_2 - 2S_1 = S$ to the right of (10.11), as expected.

11 Regularization

In section 9.4, the step to the regularized version (9.14) of the inverse matrix D^{-1} was done by citation. Here this painful gap will get closed. In the equations for Γ , (9.6), the

limit $\vec{r}' \rightarrow \vec{r}$ could not be performed, not naively at least, because according to (9.9) the Greens function comes across its pathological argument zero.

First of all, a regularisation has to respect gauge invariance, i.e. it might favour H 's rather than M 's. But there is still one more redundancy to be respected, namely the “very old” one in fixing the mapping from space \mathcal{A} to space \mathcal{M} .

11.1 Holomorphic invariance

The term could be due to KKN (*we shall refer to ... as ...*). We encountered the corresponding freedom while fixing it especially in (4.4). An arbitrary $N \times N$ matrix \bar{V} , depending on $\bar{z} = x + iy$, could have been used there in place of the inhomogeneity 1. Of course, if allowing for all these inhomogeneities, then a special element A of space \mathcal{A} is mapped to some subspace $M\bar{V}$. If M is in $\text{SL}(N, \mathbb{C})$ than $M\bar{V}$ is not. Nevertheless, changing \bar{V} leaves the A fields invariant,

$$M \rightarrow M\bar{V} \quad \leadsto \quad A = -(\partial M)M^{-1} \rightarrow -(\partial M\bar{V})\bar{V}^{-1}M^{-1} = A \quad , \quad (11.1)$$

because of $\partial\bar{V}(\bar{z}) = 0$, see (3.12). No physics depends on the special V , people on earth ($\bar{V} = 1$), moon or neptun work with to fix their mapping.

The action $S[H]$ is holomorphic invariant. $M \rightarrow M\bar{V}$ makes $H = M^\dagger M$ turning into $H \rightarrow V H \bar{V}$, where $V := \bar{V}^\dagger$ depends on only $z = x - iy$ and $\bar{\partial}V(z) = 0$. Hence, the question is, whether $S[V H \bar{V}]$ agrees with $S[H]$, and this is clearly a matter of the $S[ABC]$ relation (10.12) as prepared in the last subsection. The three terms under trace vanish all due to $\partial\bar{C} = \partial\bar{V} = 0$ and/or $\bar{\partial}A = \bar{\partial}V = 0$. The remaining unwanted terms are $S[V]$ and $S[\bar{V}]$. They vanish, as is seen next, because either V or \bar{V} suffer under the “false differentiation”. For $S_1[V]$ this is seen in (9.27) directly. Concerning $S_2[V]$ consider the square bracket in (10.2) :

$$\begin{aligned} [X_1 X_2 - X_2 X_1] &= V^{-1} V_{11} V^{-1} V_{22} - V^{-1} V_{22} V^{-1} V_{11} \\ &= \frac{2}{i} V^{-1} (\bar{\partial} V) V^{-1} \partial V - \frac{2}{i} V^{-1} (\partial V) V^{-1} \bar{\partial} V = 0 \quad , \end{aligned} \quad (11.2)$$

since $\bar{\partial}V(z) = 0$. Similarly $S_2[\bar{V}]$ is shown to vanish. Thus, $S[V H \bar{V}] = S[H]$, q.e.d.

The Greens function becomes a matrix. Let the people on the moon (L for Luna) represent the field A by $A = -(\partial L)L^{-1}$ and use the inhomogeneity \bar{V} when solving it for L . They write (using $G_{rr'} = -G_{r'r}$)

$$L = \bar{V} + \int' (AL)_{r'} G_{r'r} \quad \text{in place of} \quad M = 1 + \int' (AM)_{r'} G_{r'r} \quad . \quad (11.3)$$

We (living on earth) multiply our integral equation (the right one in (11.3)) by \bar{V} from the right,

$$M\bar{V} = \bar{V} + \int' (AM\bar{V})_{r'} \bar{V}_{r'}^{-1} G_{r'r} \bar{V}_r \quad ,$$

and recognize the two manipulations in parallel

$$\begin{aligned} M &\rightarrow M \bar{V} & \text{and} & & G_{rr'} &\rightarrow \bar{V}_r^{-1} G_{rr'} \bar{V}_{r'} =: \tilde{G}_{rr'} \\ \text{or } M^\dagger &\rightarrow V M^\dagger & \text{and} & & \bar{G}_{rr'} &\rightarrow V_r \bar{G}_{rr'} V_{r'}^{-1} \end{aligned} \quad (11.4)$$

by which one is beamed to Luna. In passing, $\tilde{G}_{rr'}$ is a Greens function as well : $\partial \tilde{G}_{rr'} = \delta(\vec{r} - \vec{r}')$.

Just for fun, let as also turn to neptun, where people write $A = -(\partial N)N^{-1}$ and work with inhomogeneity \bar{W} and Greens function $\bar{U}_r G_{rr'} \bar{U}_{r'}^{-1}$, say :

$$N = \bar{W} + \int' (AN)_{r'} \bar{U}_{r'} G_{rr'} \bar{U}_r^{-1} \quad .$$

Now, multiplication with \bar{U} from the right leads back to Luna equations with $L = N\bar{U}$ and inhomogeneity $\bar{V} = \bar{W}\bar{U}$. Allright.

Finally, for later use, we add the adjoint representation of $\tilde{G}_{rr'}$, as defined in (11.4) :

$$G_{rr'} \delta^{ab} \rightarrow (\bar{V}_r^{-1})^{ac} G_{rr'} (\bar{V}_{r'})^{cb} =: (\tilde{G}_{rr'})^{ab} \quad . \quad (11.5)$$

To derive this, simply repeat the steps leading from (11.3) to (11.4), but in adjoint representation.

The integration measure $d\mu(\mathcal{H})$ is holomorphic invariant. According to KKN this is *easily checked*. To do so, we look at (8.10), i.e. at $ds_{\mathcal{H}}^2 = 2 \int \text{Tr}(H^{-1} \delta H H^{-1} \delta H)$. While varying H the inhomogeneity \bar{V} has to be kept fixed: one either lives on moon or neptun or just here. But under $H \rightarrow V H \bar{V}$ and $\delta H \rightarrow V \delta H \bar{V}$ the metrics remains clearly unchanged, and so does the measure.

The operators p^a and \bar{p}^a split off a matrix. Two functional differential operators, called p^a and \bar{p}^a , are particular useful for the present task (see (11.15) below). But they enjoy still an other nice property as detailed in § 14.3. p^a and \bar{p}^a are defined through

$$\delta^a =: M_r^{ab} \int' G_{rr'} p_{r'}^b, \quad \delta^{a*} =: -(M_r^\dagger)^{ba} \int' \bar{G}_{rr'} \bar{p}_{r'}^b, \quad (11.6)$$

which is KKN's [2.34]. Remember $\delta^a = \delta_{A^a(\vec{r})}$, and note that $\bar{p}^a = -p^{a*}$. (11.6) can be solved for p by operating with $\partial_r \hat{M}^{-1}$ from the left :

$$p_r^a = \partial_r (M_r^{-1})^{ac} \delta_r^c, \quad \bar{p}_r^a = -\bar{\partial}_r (M_r^\dagger)^{ac} \delta_r^{c*}. \quad (11.7)$$

Note the position marks: ∂ or $\bar{\partial}$ act on both. Turning to Luna, the p 's change according to : folgendes :

$$p^a \rightarrow (\bar{V}^{-1})^{ab} p^b, \quad \bar{p}^a \rightarrow V^{ab} \bar{p}^b. \quad (11.8)$$

To derive this, note that A^a , or δ^a , knows of no earth-moon difference. Hence

$$\begin{aligned} p^a \rightarrow \partial 2 \text{Tr} \left(T^a \bar{V}^{-1} M^{-1} T^c M \bar{V} \right) \delta^c &= \partial 2 \text{Tr} \left(\bar{V} T^a \bar{V}^{-1} T^b \right) 2 \text{Tr} \left(T^b M^{-1} T^c M \right) \delta^c \\ &= \partial (\bar{V}^{-1})^{ab} (M^{-1})^{bc} \delta^c = (\bar{V}^{-1})^{ab} p^b. \end{aligned} \quad (11.9)$$

The step within the first line used *concatenation* in backward direction.

The Hamiltonian density is holomorphic invariant. This is trivial for the potential term in (2.10), because it is made up of the holomorphically insensitive A fields. The same is true for the kinetic energy density $\mathcal{T} = -\frac{e^2}{2}\delta^{a*}\delta^a$ — sorry for looking forward to (12.2) below. But, using (11.6), \mathcal{T} may be also written as

$$\mathcal{T} = -\frac{e^2}{2}\delta^{a*}\delta^a = \frac{e^2}{2}H^{ab}(\overline{Gp})^a(Gp)^b, \quad (11.10)$$

where $H^{ab} = 2\text{Tr}(T^a H T^b H^{-1})$. While checking the invariance with the expression to the right, we enjoy the harmony among the previous equations (11.5) to (11.8) :

$$(Gp)_r^b := \int' G_{rr'} p_{r'}^b \rightarrow \int' (\overline{V}_r^{-1} G_{rr'} \overline{V}_{r'})^{bc} (\overline{V}_{r'}^{-1})^{cd} p_{r'}^d = (\overline{V}_r^{-1})^{bc} (Gp)_r^c, \quad (11.11)$$

and similarly $(\overline{Gp})^a \rightarrow V^{ab}(\overline{Gp})^b$. From $H \rightarrow V H \overline{V}$, and using concatenations and $V^{ac} = (V^{-1})^{ca}$, one obtains

$$\begin{aligned} H^{ab} &\rightarrow V^{ac} H^{cd} \overline{V}^{db} \quad \hookrightarrow \\ \mathcal{T} &\rightarrow \frac{e^2}{2} V^{ac} H^{cd} \overline{V}^{db} V^{ae} (\overline{Gp})^e (\overline{V}^{-1})^{bf} (Gp)^f = \mathcal{T}, \quad \text{q.e.d.} \end{aligned} \quad (11.12)$$

11.2 Point splitting

Usually, the infinites of a field theory are recognized and regularized as UV catastrophes in momentum space $(\Lambda, M, d - \varepsilon)$. If this is done in real space, we expect that some short distances must be washed out, as e.g. when going from the delta function to one of its representations. KKN's smooth delta function is

$$\sigma(\vec{r}) := \frac{1}{\pi\eta} e^{-r^2/\eta}, \quad \int d^2r \sigma(\vec{r}) = \frac{1}{\pi} \int e^{-r^2} = 1 \quad (11.13)$$

The inverse momentum cutoff η is small but non-zero. Compared to this the small parameter ε in the Greens function (4.2) is an icecold $+0$.

In a manner that respects holomorphic properties, the smoothening σ is first buildt in the operators p . KKN then form $(Gp)^a$ to read off from it the fate of G itself. The regularized Greens function, thanks to some labour of evaluation, exhibits the desired finite value in the coincidence limit $\vec{r}' \rightarrow \vec{r}$.

Things start with a notation. Let H , if considered as a function of z and \bar{z} , be denoted by K :

$$H^{ab}(\vec{r}) =: K^{ab}(z, \bar{z}) \rightarrow V^{ac}(z) K^{cd}(z, \bar{z}) \overline{V}^{db}(\bar{z}). \quad (11.14)$$

For $\eta \rightarrow 0$, i.e. $\sigma_{rr'} \rightarrow \delta_{rr'}$, the two expressions

$$\begin{aligned} p_{\text{reg}}^a &:= \int' \sigma_{rr'} [K^{-1}(z', \bar{z}) K(z', \bar{z}')]^{ab} p_{r'}^b \\ \bar{p}_{\text{reg}}^a &:= \int' \sigma_{rr'} [K(z, \bar{z}') K^{-1}(z', \bar{z}')]^{ab} \bar{p}_{r'}^b \end{aligned} \quad (11.15)$$

clearly turn into p^a and \bar{p}^a , respectively. But even for $\sigma_{rr'} \neq \delta_{rr'}$ they have the right holomorphic behaviour (11.8). This is seen by means of (11.14): $[K^{-1}(z', \bar{z})]^{ac} \rightarrow [\bar{V}^{-1}(\bar{z})K^{-1}(\dots)V(z')]^{ac}$, and the right V matrix at unprimed argument survives at the left end. Anyhow clever.

As announced, the replacement $p^a \rightarrow p_{\text{reg}}^a$ is now performed in $(Gp)^a$,

$$\int' G_{rr'} p_{r'}^a \rightarrow \int' G_{rr'} \int'' \sigma_{r'r''} [K^{-1}(z'', \bar{z}') K(z'', \bar{z}'')]^{ab} p_{r''}^b =: \int'' \mathcal{G}_{rr''}^{ab} p_{r''}^b, \quad (11.16)$$

to read off the regularized Greensfunktion \mathcal{G} as

$$\mathcal{G}_{rr''}^{ab} = \int' G_{rr'} \sigma_{r'r''} [K^{-1}(z'', \bar{z}') K(z'', \bar{z}'')]^{ab} =: \int' G_{rr'} \sigma_{r'r''} f^{ab}(\bar{z}') \quad (11.17)$$

Denoting $[K^{-1}(z'', \bar{z}') K(z'', \bar{z}'')]^{ab}$ by $f^{ab}(\bar{z}')$, we suppress the variables irrelevant in the next subsection. But let $f^{ab}(\bar{z}'') = \delta^{ab}$ be kept in mind.

11.3 Performing the integration in \mathcal{G}

At first glance, it appears absurd to perform the $d^2 r'$ integration in (11.17), because, though G and σ are known functions, we cannot specify $f(\bar{z})$. The integration is nevertheless possible — thanks to the fact that G is a Greens function of ∂ .

There is a test, the result will have to pass. According to (11.17) it is

$$\partial \mathcal{G}_{rr''}^{ab} = \sigma_{rr''} f^{ab}(\bar{z}) = \frac{1}{\pi \eta} e^{-(\vec{r} - \vec{r}'')^2 / \eta} f^{ab}(\bar{z}) \quad (11.18)$$

and after evaluation this must be still valid, of course. The reader might look forward to (11.26) and do the test right now.

It is convenient to introduce an integral S of σ :

$$\partial S(\vec{r}) = \sigma(\vec{r}) \quad , \quad \text{ansatz } S = \frac{z}{r^2} h(r) \quad \hookrightarrow \quad S(\vec{r}) = \frac{1}{\pi \bar{z}} \left(1 - e^{-r^2/\eta}\right) \quad (11.19)$$

where a constant of integration was chosen to be 1 at will. By means of S , and omitting the indices a, b for brevity, we may write

$$\mathcal{G}_{rr''} = \int' G_{rr'} \{ \partial' S(\vec{r}' - \vec{r}'') \} f(\bar{z}') = \int' \partial' (GSf) - \int' (\partial' G) S f =: \mathcal{G}_1 + \mathcal{G}_2 \quad (11.20)$$

since $\partial' f = 0$. The second part is readily evaluated:

$$\begin{aligned} \mathcal{G}_2 &= S(\vec{r} - \vec{r}'') f(\bar{z}) = \frac{1}{\pi} \frac{1}{\bar{z} - \bar{z}''} \left(1 - e^{(\vec{r} - \vec{r}'')^2 / \eta}\right) f(\bar{z}) \\ &= G_{rr''} \left(1 - e^{(\vec{r} - \vec{r}'')^2 / \eta}\right) f(\bar{z}) \quad (11.21) \end{aligned}$$

The use of G in the last line was allowed due to the zero of the round bracket at the pole of G . The other part may be written as

$$\mathcal{G}_1 = \int' \partial' G_{rr'} \frac{1}{\pi(\bar{z}' - \bar{z}'')} (1 - e^{-\dots}) f(\bar{z}') = \frac{-1}{\pi(\bar{z} - \bar{z}'')} f(\bar{z}) + \mathcal{G}_{1e} \quad (11.22)$$

with

$$\mathcal{G}_{1e} = - \int d^2 r' \partial' \left[\frac{1}{\pi} \frac{z - z'}{(z - z')(\bar{z} - \bar{z}') + \varepsilon^2} \frac{1}{\pi} \frac{1}{\bar{z}' - \bar{z}''} e^{-(\vec{r}' - \vec{r}'')^2/\eta} f(\bar{z}') \right] , \quad (11.23)$$

For the first step in analysing this expression, the reader might realize that the limit $\varepsilon \rightarrow 0$ may be performed here. The second step is the shift $\vec{r}' \rightarrow \vec{r}' + \vec{r}''$ of the integration variable. The third step introduces the variables u, v by

$$x' = \frac{1}{2}(u - iv) , \quad y' = \frac{1}{2}(-iu + v) , \quad \text{Jacobian} = \frac{1}{2} , \quad \bar{z}' = u , \quad z' = -iv ,$$

and leads to

$$\mathcal{G}_{1e} = -\frac{i}{2} \frac{1}{\pi} \int du' \frac{f(u' + u'')}{u - u'' - u'} \frac{1}{\pi u'} \int dv' \partial_{v'} e^{i \frac{u'}{\eta} v'} = \frac{1}{\pi} \frac{1}{\bar{z} - \bar{z}''} f(\bar{z}'') . \quad (11.24)$$

To realize this, perform $\partial_{v'}$, obtain $\delta(u')$ from the v' integration and return to $u'' = \bar{z}''$. Using (11.24) in (11.22) we arrive at

$$\mathcal{G}_1 = \frac{1}{\pi} \frac{1}{\bar{z} - \bar{z}''} [f(\bar{z}'') - f(\bar{z})] = G_{rr''} [f(\bar{z}'') - f(\bar{z})] \quad (11.25)$$

and together with (11.21) at the final result

$$\mathcal{G}_{rr''}^{ab} = G_{rr''} \left(\delta^{ab} - e^{-(\vec{r} - \vec{r}'')^2/\eta} f^{ab}(\bar{z}) \right) . \quad (11.26)$$

(11.26) is [3.8]. It was obtained with the special embedding (11.13) of the delta function, but we bet that any other delta representation would lead to the same result.

11.4 The coincidence limit

After all, the limit $\vec{r}' \rightarrow \vec{r}$ can now be performed without troubles. We look back at \mathcal{G}_2 , (11.21), and realize that the round bracket vanishes $\sim (z - z'')(\bar{z} - \bar{z}'')$, i.e. faster than the denominator. In short, in the coincidence limit \mathcal{G}_2 turns to zero. For \mathcal{G}_1 , the inner expression of (11.25) shows a differential quotient:

$$\begin{aligned} \mathcal{G}_{rr}^{ab} &= \mathcal{G}_{1rr}^{ab} = -\frac{1}{\pi} \frac{\Downarrow}{\bar{\partial}} \left[K^{-1}(z, \frac{\Downarrow}{\bar{z}}) K(z, \bar{z}) \right]^{ab} = \frac{1}{\pi} [H^{-1} \bar{\partial} H]^{ab} \\ &= \frac{1}{\pi} \left[M^{-1} M^{\dagger -1} \left((\bar{\partial} M^{\dagger}) M + M^{\dagger} \bar{\partial} M \right) \right]^{ab} = \frac{-1}{\pi} \left[M^{-1} \left(A^{\dagger} - (\bar{\partial} M) M^{-1} \right) M \right]^{ab} . \end{aligned} \quad (11.27)$$

(11.27) is [3.10]. In regularizing (9.9), just G has to be replaced by \mathcal{G} :

$$\left[D_{\text{reg}}^{-1} \right]_{rr}^{ab} = M^{ac} \mathcal{G}_{rr}^{cd} (M^{-1})^{db} = -\frac{1}{\pi} [A^{\dagger} - (\bar{\partial} M) M^{-1}]^{ab} , \quad \text{q.e.d.} \quad (11.28)$$

and hooray, because this is the desired result (9.14).

12 Kinetic energy , mass gap and CFT

Anywhere, far behind, we had noticed the classical Hamiltonian density of the 2+1 D YM system, (2.8) to (2.10) :

$$\mathcal{L} = \frac{1}{2e^2} \dot{A}_j^a \dot{A}_j^a - \mathcal{V} \quad , \quad \Pi_j^a = \frac{1}{e^2} \dot{A}_j^a \quad , \quad \mathcal{H} = \frac{e^2}{2} \Pi_j^a \Pi_j^a + \mathcal{V} \quad (12.1)$$

with $\mathcal{V} := \frac{1}{2e^2} B^a B^a$ and $B^a = \partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c$. Were there no gauge freedom, the quantum mechanics would be initiated by $\Pi_j^a \rightarrow (1/i) \delta_{A_j^a}$:

$$\mathcal{H} = \mathcal{T} + \mathcal{V} \quad \text{with} \quad \mathcal{T} = -\frac{e^2}{2} \delta_{A_j^a} \delta_{A_j^a} = -\frac{e^2}{2} \delta^{a*} \delta^a \quad (12.2)$$

as already announced in (11.10). The question, how \mathcal{T} can be restricted to the physical subspace \mathcal{C} , has a rather simple answer : apply \mathcal{T} to functionals of only H ! KKN make some efforts in constructing the *Laplacian on \mathcal{C}* by splitting off the unphysical part from p^a . The use of these details will be realized in § 14.4. But they are irrelevant for the particular question of reducing \mathcal{T} . It restricts itself by working with in the space \mathcal{C} . Thereby the above “would be” turns into “is”, automatically.

12.1 $\mathcal{T} J^a = m J^a$

Let \mathcal{T} be applied to special functionals $\psi[H]$. Which ones ? At this point we jump and servile follow the text of KKN around [2.28]. There one finds the statement, which is crucial and is as interesting as (at first glance) mysterious :

To be normalizable , the ψ 's must be functionals $\psi[J^a]$ of the “currents”

$$J^a = \frac{2N}{\pi} \text{Tr} \left(T^a (\partial H) H^{-1} \right) \quad . \quad (12.3)$$

This is the decomposition of $\psi[H]$'s expected at the end of § 10.2. The statement means that the dependence on H must be indirect as shown, otherwise the norm $\int |\psi|^2$ diverges. The objects J^a are indeed the currents of the WZW action S as detailed at the end of § 12.2. To justify (12.3), apparently (see [1], this time really), there are several possibilities. Probably, conformal field theory (CFT) provides with the most convincing reasoning. But let CFT be outside of the scope of the present treatise (apart from the next subsection). Some weaker argument in favour of (12.3) rests on fact that the Wilson loop, hence any (normalizable) physics, can be shown to depend on only J . This will be detailed in § 16.5.

For ending up part I, consider the most simple functional of J , namely [4]

$$\psi_{\text{sp}} [J] := \int c^a(\vec{r}) J^a(\vec{r}) \quad (12.4)$$

(with arbitrary c number functions $c^a(\vec{r})$) and apply $\int \mathcal{T}$ to it. Using

$$\begin{aligned} J^a(\vec{r}) &= \frac{N}{\pi} 2 \operatorname{Tr} (T^a [(\partial M^\dagger) M^{\dagger-1} + M^\dagger i T^c A^c M^{\dagger-1}]) \\ &\curvearrowright \delta_r^a J_{r'}^d = i \frac{N}{\pi} (M_r^\dagger)^{da} \delta(\vec{r} - \vec{r}') \end{aligned} \quad (12.5)$$

we are led to the following second line. But for the details (*), (**) (to be commented afterwards) we are much to excited :

$$\begin{aligned} \mathbf{T} \psi_{\text{sp}} &= -\frac{e^2}{2} \int \delta_r^{a*} \delta_r^a \int' c_{r'}^d J_{r'}^d \\ &= -\frac{e^2}{2} \int \delta_r^{a*} \frac{N}{\pi} i c_{r'}^d (M_r^\dagger)^{da} = -i \frac{e^2 N}{2\pi} \int c^d [\delta_{r'}^{a*} (M_r^\dagger)^{da}]_{\vec{r}' \rightarrow \vec{r}} \\ &= -i \frac{e^2 N}{2\pi} \int c^d \left[(M_{r'}^\dagger)^{db} f^{bae} (D_{rr'}^{*-1})^{ae} \right]_{\vec{r}' \rightarrow \vec{r}} \quad (*) \\ &= -i \frac{e^2 N}{2\pi} \int c^d (M^\dagger)^{db} \frac{iN}{\pi} 2 \operatorname{Tr} (T^b [M^{\dagger-1} \partial M^\dagger + (\partial M) M^{-1}]) \quad (**) \\ &= \frac{e^2 N}{2\pi} \int c^d \left[\frac{N}{\pi} 2 \operatorname{Tr} (T^d (\partial H) H^{-1}) \right] \\ &= \frac{e^2 N}{2\pi} \int c^d J^d = m \psi_{\text{sp}} \quad , \quad m := \frac{e^2 N}{2\pi} . \end{aligned} \quad (12.6)$$

Special eigenfunctions of \mathbf{T} are found. Due to the arbitrariness of the weight $c^a(\vec{r})$ the corresponding eigenvalue m is infinitely degenerated.

In the third line (*), one recognized the same coincidence limit as already in (9.6). Moreover, the factors $f^{\cdots} D^{*-1}$ in the square bracket are $\delta^{b*} \Gamma$, i.e. they can be replaced by the regularized expression (9.20) directly. This explains the step from line (*) to line (**). Then, by just a *concatenation* one reaches the next-to-last line. There remains the problem of verifying the step from the second to the third line. It states the equality of two square brackets. For simplicity, consider the c.c. of the first one. $\delta_{r'}^a \hat{M}_r$ must derive from $0 = (\partial + \hat{A}) \hat{M}$, anyhow (see (9.12) and remember $A^{ab} = A^\bullet f^{a\bullet b}$). We apply $\delta_{r'}^a$, use (7.13) and obtain

$$\int'' D_{rr'}^{cb} \delta_{r'}^a M_{r''}^{bd} = \delta_{rr'} f^{aeb} M_{r'}^{bd} \quad \curvearrowright \quad \delta_{r'}^a M_r^{cd} = (D_{rr'}^{-1})^{ce} f^{aeb} M_{r'}^{bd} . \quad (12.7)$$

One may set $a = c$ here and sum over a . Since $M^{db*} = (M^\dagger)^{bd}$ we may even add the c.c. with ease :

$$\delta_{r'}^a M_r^{ad} = (D_{rr'}^{-1})^{ae} f^{aeb} M_{r'}^{bd} \quad , \quad \delta_{r'}^{a*} (M_r^\dagger)^{da} = (M_{r'}^\dagger)^{db} f^{bae} (D_{rr'}^{*-1})^{ae} , \quad (12.8)$$

q.e.d. But there is danger to anyone going his own way. Instead of (12.7) he could derive the very similar relation

$$\delta_{r'}^a M_r^{cd} = (D_{rr'}^{-1})^{ea} f^{ecb} M_r^{bd} . \quad (12.9)$$

Now, by $c = a$ and summation, he obtains (12.7) only nearly, because M^{bd} carries the index r (not r'). Hence, he condemns (12.6) and is searching for his or our error till

tomorrow. The answer: both are right. (12.7) and (12.9) can be derived from each other by means of (9.9). In the special combination (12.8), one is allowed to switch the M -index from \vec{r}' to \vec{r} .

12.2 $S[H]$ is conformally invariant

Already since section 2 we live in a 2D Euclidian world. This is the case in which the call for conformal invariance is particular restrictive [15, 16, 17], because in the coordinate transformation

$$\begin{aligned} z \rightarrow f(z) \quad , \quad \text{i.e.} \quad z_{\text{new}} = u(x, y) - iv(x, y) \stackrel{!}{=} f(x - iy) \quad (12.10) \\ \curvearrowright \quad u_{,x} = v_{,y} \quad , \quad u_{,y} = -v_{,x} \end{aligned}$$

the function $f(z)$ remains arbitrary. All these nice facts as e.g. the conservation of angles (explaining the word “conform”) under the transformation (12.10), or the change of flat metrics from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $f' f'^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ [16], may be bypassed here.

As (12.10) is a pure coordinate transformation, i.e. the values $H(z, \bar{z})$ of the fields remain unchanged. Infinitesimally ($\varepsilon \rightarrow 0$, $g(z)$ arbitrary) this means

$$\begin{aligned} z \rightarrow z' &= z + \varepsilon g(z) \\ H'(z', \bar{z}') &= H(z, \bar{z}) = H(z' - \varepsilon g, \bar{z}' - \varepsilon \bar{g}) \\ &= H(z', \bar{z}') - \varepsilon g \partial H - \varepsilon \bar{g} \bar{\partial} H =: H(z', \bar{z}') + \delta H \quad . \end{aligned} \quad (12.11)$$

To demonstrate the conformal invariance of $S[H]$ we have to show that

$$\begin{aligned} S' - S &= \int d^2 r' \mathcal{L} \{ H'(z', \bar{z}'), \partial', \bar{\partial}' \} - S = \int d^2 r \mathcal{L} \{ H + \delta H, \partial, \bar{\partial} \} - S \\ &= S [H + \delta H] - S [H] = \delta S \quad \text{with especially} \quad \delta H = -\varepsilon g \partial H - \varepsilon \bar{g} \bar{\partial} H \end{aligned} \quad (12.12)$$

does vanish. For a variation δS of S — due to whatsoever δH — we have the ready formula (9.33) :

$$\delta S = \frac{1}{\pi} \int \text{Tr} \left(\varphi \partial \bar{X} \right) \quad \text{with} \quad \{ \varphi, X, \bar{X} \} = H^{-1} \{ \delta, \partial, \bar{\partial} \} H \quad . \quad (9.33)$$

Just δH from (12.12), i.e. $\varphi = H^{-1} \delta H = -\varepsilon g X - \bar{g} \bar{X}$, has to be inserted here :

$$\begin{aligned} \delta_{\text{confo}} S &= -\frac{\varepsilon}{\pi} \int \left[g \text{Tr} (X \partial \bar{X}) + \bar{g} \text{Tr} (\bar{X} \partial \bar{X}) \right] \\ &= -\frac{\varepsilon}{\pi} \int \left[g \text{Tr} (X [\bar{\partial} X + \bar{X} X - X \bar{X}]) + \bar{g} \text{Tr} (\bar{X} \partial \bar{X}) \right] \\ &= -\frac{\varepsilon}{2\pi} \int \left[\bar{\partial} g \text{Tr} (X X) + \partial \bar{g} \text{Tr} (\bar{X} \bar{X}) \right] = 0 \quad , \quad \text{q.e.d.} \end{aligned} \quad (12.13)$$

In the second line, (9.34), i.e. $\bar{\partial} X + \bar{X} X = \partial \bar{X} + X \bar{X}$, was used, and $\bar{\partial} g = \partial \bar{g} = 0$ in the third. The conformal invariance of S is thus an immediate consequence of its functional

differential equations. One more reason for remembering the above δS formula (9.33) is realized next.

The WZW equations of motion follow from $\delta S = 0$ under independent variation of the n elements of the matrix H . They can be read off from (9.33) as

$$\begin{aligned} \partial \overline{X} = 0 \quad \quad \quad \text{or} \quad \quad \partial J^\dagger = 0 \quad \text{with} \quad J^\dagger := \frac{N}{\pi} H^{-1} \overline{\partial} H \\ \text{as well as} \quad \quad \overline{\partial} J = 0 \quad \text{with} \quad J := \frac{N}{\pi} (\partial H) H^{-1} \quad . \end{aligned} \quad (12.14)$$

As (12.14) exhibits equations of continuity, J and J^\dagger are matrix versions of the currents of the WZW model.

The currents J^a as given in (12.3) are now recognized to be the components $J^a = 2 \text{Tr}(T^a J)$ of its matrix version. By concatenation one obtains $J = T^a J^a$. But are we really allowed for expanding J , (12.14), into the traceless generators? For this, $(\partial H) H^{-1}$ has to be traceless, too. To verify it, O. Lechtenfeld had the right idea. Remembering (5.8) or (8.14), we may use the representation $H = \rho^2 = e^{2\vec{\omega}\vec{T}} =: e^{\vec{\varphi}\vec{T}}$ to get

$$\text{Tr}\left((\partial H) H^{-1}\right) = \text{Tr}\left(\int_0^1 ds e^{s\vec{\varphi}\vec{T}} (\partial\varphi^a) T^a e^{-s\vec{\varphi}\vec{T}}\right) = (\partial\varphi^a) \text{Tr}(T^a) = 0 \quad (12.15)$$

indeed.

Looking back to (12.14), one could ask, why not the other version J^\dagger , hence $J^{a*} = \frac{N}{\pi} 2\text{Tr}(T^a H^{-1} \overline{\partial} H)$, has been declared to be the WZW current. May one work with $\psi[J^{a*}]$ as well? Yes. Then one would derive the c.c. of the Hamiltonian of § 13.2. But decide to work with *e i t h e r* $\psi[J^a]$ *o r* $\psi[J^{a*}]$. This rule would be violated, for instance, by absorbing e^{NS} into the wave functionals.

12.3 End

Some first period of efforts ends up here. Just the first piece of physics has been grasped, namely the mass gap $m = e^2 N / (2\pi)$. But its stability under the inclusion of the potential term is not yet studied. There is still no functional Hamiltonian, no ground state wave functional, no Wilson loop, no confinement, no application to the 4D gluon plasma.

The reason for breaking the notes into two parts has a psychological background. For the first time feeling good with the matter, one finds reasons for relaxation — more than half a year, actually. So, have at least a nice weekend. First part, first run.

P A R T II

One year after ending up the first run. Meanwhile, there is a first attempt of using KKN's results for thermal 3+1 D physics [18]. In the static limit ($\omega \rightarrow 0$), the transverse selfenergy could be identified with the squared mass gap $m^2 = e^4 N^2 / (4\pi^2)$ through $e^2 = g^2 T$. In other words, we (Reinbach and Schulz) verified this strong conjecture by ruling out potential other contributions at order g^4 . Regulators for the 3D Theorie were derived from its embedding in the 4D setup. One may state that, for the static selfenergy, Lindes infinite numerical series has been summed up. The relation to 4D TFT is enforced in a recent report of Nair [19] (conference at Dubna).

In the following second part of the notes, they reach the end of [1], in essence, and continue with KKN's paper [5] : *On the vacuum wavefunction and string tension of Yang–Mills theories in (2+1) dimensions*. Equations in [5] are referred to as **{xx}**.

13 The full Hamilton in $\psi[J]$ space

In the retrospect, at least, we become aware of the most painful break in § 12, namely as we were satisfied with the first rough look at the spectrum: m . A l l about 2+1 D YM has to be worked out, and this “all” is contained in the functional Hamiltonian $\mathbf{H} = \int \mathcal{H} = \int \mathcal{T} + \int \mathcal{V}$ p l u s the associated exotic scalar product $\langle 1|2 \rangle$. The latter is taken up again in § 14. We shall continue with keeping distance to CFT. But let the Wilson–loop argument as detailed in § 16.4 give some confidence in the strange statement that the wave functionals ψ become a finite norm through the indirect dependence on H via $J^a = \frac{N}{\pi} 2\text{Tr} (T^a(\partial H)H^{-1})$.

13.1 Kinetic energy in $\psi[J]$ space

It is obvious how to overcome the extremely special wave functionals $\psi = \int c^a J^a$ of § 12.1. Apply the operator $\mathbf{T} = \int \mathcal{T} = -\frac{e^2}{2} \int \delta^{a*} \delta^a$ of the kinetic energy to a general functional $\psi[J^a]$. Thereby, \mathbf{T} will turn out as an expression in J^a and δ_{J^a} , automatically. As the symbolic line

$$\delta^* \delta \psi = \delta^* \int (\delta J) \delta_J \psi = \int (\delta^* \delta J) \delta_J \psi + \int (\delta J) \delta^* \delta_J \psi \quad (13.1)$$

shows, two terms will arise, which we denote by \mathbf{T}_1 and $(\mathbf{T}_2 + \mathbf{T}_3)$, respectively. KKN's headline of § 4 reads *An expression for \mathcal{T} in terms of currents*. But no, not “An”, the

one unique J version of \mathbf{T} is in search. We overlook all the epsilon care in KKN, find a direct way to the more aesthetic final result $[\mathbf{4.13}] \equiv \{\mathbf{17}\} \equiv (13.9)$ below, and enjoy giving details :

$$\begin{aligned} \mathbf{T}_1 \psi[J^a] &= -\frac{e^2}{2} \int_r \int_{r'} (\delta_r^{a*} \delta_r^a J_{r'}^d) \delta_{J_{r'}^d} \psi = \int_{r'} (\delta_{J_{r'}^d} \psi) \mathbf{T} J_{r'}^d \\ (12.6) : &= m \int J^a \delta_{J^a} \psi[J^a] \quad \hookrightarrow \quad \mathbf{T}_1 = m \int J^a \delta_{J^a} . \end{aligned} \quad (13.2)$$

This could have been guessed. For $(\mathbf{T}_2 + \mathbf{T}_3)$, however, such a quick tracing back is no more available, because after commuting the differentiations δ_r^{a*} and $\delta_{J_{r'}^d}$,

$$\begin{aligned} (\mathbf{T}_2 + \mathbf{T}_3) \psi[J^a] &= -\frac{e^2}{2} \int_r \int_{r'} (\delta_r^a J_{r'}^d) \delta_{J_{r'}^d} \underbrace{\delta_r^{a*} \psi[J^a]}_{\equiv \int_{r''} (\delta_{J_{r''}^c} \psi) \delta_r^{a*} J_{r''}^c} \\ &= -m i \int_r (M_r^\dagger)^{da} \delta_{J_r^d} \int_{r'} (\delta_{J_{r'}^c} \psi) \delta_r^{a*} J_{r'}^c , \end{aligned} \quad (13.3)$$

we become aware of the new combination $\delta_r^{a*} J_{r'}^c$. For the round bracket in the first line, instead, we remembered (12.5) and were led to the second line with ease.

To study the object $\delta_r^{a*} J_{r'}^c$ we turn to its c.c. and interchange the indices, for temporarily getting the familiar Greens function $G_{rr'}$ (not \bar{G}) involved. At first we obtain

$$\begin{aligned} \delta_{r'}^a J_r^{c*} &= \delta_{r'}^a \frac{N}{\pi} 2 \text{Tr} (T^c H^{-1} \bar{\partial} H) \\ &= \frac{N}{\pi} 2 \text{Tr} \left(T^c \left[-H^{-1} (\delta_{r'}^a H) H^{-1} \bar{\partial} H + H^{-1} \bar{\partial} (\delta_{r'}^a H) \right] \right) . \end{aligned} \quad (13.4)$$

Remember that $\delta_{r'}^a H = M^\dagger \delta_{r'}^a M$. To continue with $\delta_{r'}^a M$, only the adjoint version (12.8) is available in the previous. So, let us obtain $\delta_{r'}^a M$ anew, namely from $0 = (\partial + A)M$. Applying $\delta_{r'}^a$ leads to the differential equation $(\partial + A)(\delta_{r'}^a M) = iT^a M_r \delta_{rr'}$. Its solution is $\delta_{r'}^a M = M_r G_{rr'} M_{r'}^{-1} iT^a M_{r'}$ and is easily verified (do not sum over \vec{r} or \vec{r}'). This gives

$$\delta_{r'}^a H = i H G_{rr'} M_{r'}^{-1} T^a M_{r'} = i H G_{rr'} T^b 2 \text{Tr} (T^b M_{r'}^{-1} T^a M_{r'}) = H T^b i G_{rr'} M_{r'}^{ab} \quad (13.5)$$

under suppression of un-primed r indices, as already in (13.4). Hence, the last $\bar{\partial}$ in (13.4) will produce two terms, one with $\bar{\partial} H$ and one with $\bar{\partial} G_{rr'}$:

$$\begin{aligned} \delta_{r'}^a J_r^{c*} &= \frac{N}{\pi} 2 \text{Tr} \left([T^b, T^c] H^{-1} \bar{\partial} H \right) i G_{rr'} M_{r'}^{ab} + \frac{N}{\pi} \delta^{bc} i \bar{\partial} G_{rr'} M_{r'}^{ab} \\ &= M_{r'}^{ab} \left[-\frac{1}{\pi(\bar{z} - \bar{z}')} f^{bce} J_r^{e*} - \frac{N}{\pi} i \delta^{bc} \frac{1}{\pi(\bar{z} - \bar{z}')^2} \right] . \end{aligned} \quad (13.6)$$

Somewhat dangerous happened in turning to the second line by inserting the un-embedded Greens function $G_{rr'} = 1/[\pi(\bar{z} - \bar{z}')] (and even operating with $\bar{\partial} = \partial_{\bar{z}}$ on it). Ultimately, the corresponding doubts will have gone in the Fourier subsection § 13.3. We now return to what we need in (13.3) :$

$$\delta_r^{a*} J_{r'}^c = (M_r^{\dagger-1})^{ab} \left[\frac{1}{\pi(z - z')} f^{bce} J_{r'}^e + \frac{N}{\pi} i \delta^{bc} \frac{1}{\pi(z - z')^2} \right] , \quad (13.7)$$

because $M^{ab*} = (M^\dagger)^{ba} = (M^{\dagger-1})^{ab}$. Clearly, the objects M^\dagger will recombine in (13.3). There we are. Just the unspecified $\psi[J]$ now might be omitted on both sides :

$$\mathbf{T}_2 + \mathbf{T}_3 = m \int_r \int_{r'} \delta_{J_r^b} \left[\frac{N}{\pi^2} \frac{\delta^{bc}}{(z - z')^2} - \frac{i}{\pi} \frac{f^{bce}}{(z - z')} J_{r'}^c \right] \delta_{J_{r'}^c} \quad . \quad (13.8)$$

Doesn't it smell to CFT a bit? Due to the f antisymmetry the functional derivative $\delta_{J_r^b}$ may be shifted through the square bracket. Thus, the functional operator of the kinetic energy in the space of $\psi[J]$ is given by

$$\begin{aligned} \mathbf{T} &= \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 \\ &= m \int d^2r J^a \delta_{J^a} + m \int d^2r \int d^2r' \left\{ \frac{N}{\pi^2} \frac{\delta^{ab}}{(z - z')^2} - \frac{i}{\pi} \frac{f^{abc}}{(z - z')} J_{r'}^c \right\} \delta_{J_r^a} \delta_{J_{r'}^b} \quad . \quad (13.9) \end{aligned}$$

As already announced, the seemingly dangerously singular denominators will turn out to be harmless in § 13.3. (13.9) is **{ 17 }**. Doesn't the last term vanish due to f antisymmetry? No, because \vec{r} , \vec{r}' would have to be interchanged, too, under change of sign of the denominator.

13.2 Potential \mathbf{V} in $\psi[J]$ space

The term $\mathbf{V} = \int \mathcal{V}$ with $\mathcal{V} = \frac{1}{2e^2} B^a B^a$ and $B^a = \partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c$ was noticed long ago in (2.9) and nearly forgotten afterwards. It is real and contains no functional derivatives. So, as it stands, it might turn out to be a functional of only J . Via $B^a \rightarrow A \rightarrow M \rightarrow H \rightarrow J$, where each arrow stands for “eliminate in favour of”, it gets thrilling with the last two arrows. For the first, one can clearly write

$$\begin{aligned} \mathcal{V} &= \frac{1}{2e^2} B^a B^a = \frac{1}{e^2} \text{Tr}(BB) \quad , \quad B := B^a T^a = B^\dagger \\ B &= T^a (\partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c) = i\partial_1 A_2 - i\partial_2 A_1 + [A_1, A_2] \\ &= 2(\bar{\partial}A - \partial\bar{A} + [\bar{A}, A]) = 2\left\{ (\bar{\partial} + \bar{A})A - (\partial + A)\bar{A} \right\} \quad , \quad (13.10) \end{aligned}$$

where $\bar{A} := (A_1 - iA_2)/2 = -A^\dagger$. Matrices M come into play by $A = -(\partial M)M^{-1}$ and $\bar{A} = -A^\dagger = M^{\dagger-1}\bar{\partial}M^\dagger$. Here, it would be easier to go from the result (13.12) in backward direction. But we like inventing so much :

$$\begin{aligned} (\bar{\partial} + \bar{A})A &= \bar{\partial}A + M^{\dagger-1}(\bar{\partial}M^\dagger)A = M^{\dagger-1}\bar{\partial}M^\dagger A = -M^{\dagger-1}\bar{\partial}M^\dagger(\partial M)M^{-1} \\ &= -M^{\dagger-1}\bar{\partial}(\partial H)H^{-1}M^\dagger + M^{\dagger-1}\bar{\partial}\partial M^\dagger \\ &= -M^{\dagger-1}[\bar{\partial}(\partial H)H^{-1}]M^\dagger + \underbrace{M^{\dagger-1}[\partial - (\partial H)H^{-1}]\bar{\partial}M^\dagger}_{\text{second term}} \\ \text{second term} &= M^{\dagger-1}[\partial - (\partial M^\dagger)M^{\dagger-1} - M^\dagger(\partial M)M^{-1}M^{\dagger-1}]M^\dagger\bar{A} \\ &= (\partial + A)\bar{A} \quad . \quad (13.11) \end{aligned}$$

The field B is now seen to be a sandwich with a delicious H matter in between,

$$B = 2\left\{ (\bar{\partial} + \bar{A})A - (\partial + A)\bar{A} \right\} = -2M^{\dagger-1}[\bar{\partial}(\partial H)H^{-1}]M^\dagger \quad , \quad (13.12)$$

and the dry bred is devoured by the trace :

$$\begin{aligned}
\mathcal{V} &= \frac{4}{e^2} \text{Tr} \left([\bar{\partial}(\partial H)H^{-1}] [\bar{\partial}(\partial H)H^{-1}] \right) \\
&= \frac{2}{e^2} 2 \text{Tr} \left(T^a [\bar{\partial}(\partial H)H^{-1}] \right) 2 \text{Tr} \left(T^a [\bar{\partial}(\partial H)H^{-1}] \right) \\
&= \frac{2}{e^2} \left(\bar{\partial} \frac{\pi}{N} J^a \right) \bar{\partial} \frac{\pi}{N} J^a = \frac{\pi}{m N} (\bar{\partial} J^a) \bar{\partial} J^a .
\end{aligned} \tag{13.13}$$

De-concatenation has led to the second line. (13.13) is [2.45] $\equiv \{18\}$. Other than the kinetic term, the potential energy density goes as $\sim 1/m$. Remember that \mathcal{V} is real. Had we started from $\mathcal{V} = \text{Tr}(B^\dagger B^\dagger)$, then $\mathcal{V} = \frac{\pi}{mN} (\partial J^{a*}) \partial J^{a*}$ would have been obtained, which is valid too. On the other hand, $\partial J^{a*} \neq \bar{\partial} J^a$. In fact, for \mathcal{V} to be real, the summation over a is required, or the trace in the above first line, respectively.

Thus, the functional Hamiltonian \mathbf{H} in the space of $\psi[J]$ is given by

$$\begin{aligned}
\mathbf{H} = \mathbf{T} + \mathbf{V} &= m \int d^2 r J^a \delta_{J^a} \\
&+ m \int d^2 r \int d^2 r' \left\{ \frac{N}{\pi^2} \frac{\delta^{ab}}{(z - z')^2} - \frac{i}{\pi} \frac{f^{abc}}{(z - z')} J_{r'}^c \right\} \delta_{J_r^a} \delta_{J_{r'}^b} \\
&+ \frac{\pi}{m N} \int d^2 r (\bar{\partial} J^a) \bar{\partial} J^a
\end{aligned}$$

(13.14)

By a framed picture one is invited contemplating. What is seen here? Keep your thumb over the f^{abc} term. Then, a quadratic form of variables J and derivatives δ_J is noticed. Omitting integrals and color indices, (13.14) would be a harmonic oscillator involving a $x\partial_x$ term. But the latter could be removed by a gauge transformation $e^{\alpha x^2}$. Remove the thumb. Now, the framed (13.14) shows the Hamiltonian of a system of infinitely many (field theory!) coupled oscillators with the f^{abc} term as its only nonlinearity. The f term “rotates with strength J ”. One also realizes how \mathbf{H} reduces to the first term, if it is applied to a single $J^a(\vec{r})$ and if by e.g. $m \rightarrow \infty$ the last term is removed. $\mathbf{T}J^a = mJ^a$ was really special.

Some partial decoupling of the oscillators may be expected from Fourier transformation : § 13.3. One may also suspect that (13.14) could need some normal ordering prescription due to oscillator zero-point energies summed up : § 15.

13.3 Fourier transform

The following longing for harmony goes a bit ahead over the KKN material. Fourier transformation

$$J^a(\vec{r}) = \int \frac{d^2 k}{(2\pi)^2} e^{i\vec{k}\vec{r}} \tilde{J}^a(\vec{k}) \quad , \quad \tilde{J}^a(\vec{k}) = \int d^2 r e^{-i\vec{k}\vec{r}} J^a(\vec{r}) \quad , \tag{13.15}$$

must be anyhow good to the above Hamiltonian, in particular to the convolution integral ($N\delta^{ab}$ term). With view to the potential term, even the field

$$I^a(\vec{r}) := \bar{\partial} J^a(\vec{r}) \quad , \quad \tilde{J}^a(\vec{k}) = \frac{2}{i} \frac{1}{k_1 - ik_2} \tilde{I}^a(\vec{k}) \quad , \quad (13.16)$$

comes into mind, where the second relation rests on $\tilde{I}^a(\vec{k}) = \int d^2r \, e^{-i\vec{k}\vec{r}} \bar{\partial} \int \frac{d^2q}{(2\pi)^2} e^{i\vec{q}\vec{r}} \tilde{J}^a(\vec{q}) = \frac{i}{2}(k_1 - ik_2) \tilde{J}^a(\vec{k})$. Using (13.15) (right relation) the chain rule of differentiation reads

$$\delta_{J^a(\vec{r})} = \int d^2k \left(\delta_{J^a(\vec{r})} J^b(\vec{k}) \right) \delta_{\tilde{J}^b(\vec{k})} = \int d^2k \, e^{-i\vec{k}\vec{r}} \delta_{\tilde{J}^a(\vec{k})} \quad . \quad (13.17)$$

The “wrong” sign is OK, and the “missing of π ’s” as well. The factor between \tilde{J} and \tilde{I} drops out in $\tilde{J}\delta_{\tilde{J}}$. Hence, so far, two of the four H terms can be booked down :

$$\mathbf{T}_1 = m \int d^2k \, \tilde{I}^a(\vec{k}) \delta_{\tilde{I}^a(\vec{k})} \quad \text{and} \quad \mathbf{V} = \frac{1}{4\pi mN} \int d^2k \, \tilde{I}^a(\vec{k}) \tilde{I}^a(-\vec{k}) \quad . \quad (13.18)$$

The convolution integral

$$\begin{aligned} \mathbf{T}_2 &= \frac{mN}{\pi^2} \int_r \int_{r'} \delta_{J_r^a} \frac{1}{(z - z')^2} \delta_{J_{r'}^a} = 4mN \int d^2k \, \tilde{\omega}(\vec{k}) \delta_{\tilde{J}^a(\vec{k})} \delta_{\tilde{J}^a(-\vec{k})} \\ &= mN \int d^2k \, \tilde{\omega}(\vec{k}) (k_1 - ik_2)^2 \delta_{\tilde{I}^a(\vec{k})} \delta_{\tilde{I}^a(-\vec{k})} \end{aligned} \quad (13.19)$$

involves the Fourier transform $\tilde{\omega}$ of $1/z^2$, and the awkward term \mathbf{T}_3 is infected with the Fourier transform $\tilde{\kappa}$ of $1/z$:

$$\begin{aligned} \mathbf{T}_3 &= \dots \text{ (some labour) } = -i \frac{m}{\pi} f^{abc} \int d^2q \int d^2p \, \tilde{\kappa}(\vec{q}) \tilde{J}^c(\vec{p} + \vec{q}) \delta_{\tilde{J}^a(\vec{q})} \delta_{\tilde{J}^b(\vec{p})} \\ &= \frac{m}{2\pi} f^{abc} \int d^2q \int d^2p \, \tilde{\kappa}(\vec{q}) \frac{(q_1 - iq_2)(p_1 - ip_2)}{(q_1 - iq_2) + (p_1 - ip_2)} \tilde{I}^c(\vec{p} + \vec{q}) \delta_{\tilde{I}^a(\vec{q})} \delta_{\tilde{I}^b(\vec{p})} \quad . \end{aligned} \quad (13.20)$$

To complete the Fourier version, there remains some integration work on $\tilde{\omega}$ and $\tilde{\kappa}$. A reader, who likes pitfalls with contour integrations, might evaluate $\tilde{\omega}$ in kartesian coordinates. But the harmless way uses polar coordinates (even for \vec{k} by writing it as $[k \cos(\alpha), k \sin(\alpha)]$) and shows the absence of a singularity immediately :

$$\begin{aligned} \tilde{\omega}(\vec{k}) &= \int d^2r \, \frac{1}{z^2} e^{-i\vec{k}\vec{r}} = \int_0^\infty dr \, r \int_{(2\pi)} d\varphi \, \frac{e^{i2\varphi}}{r^2} e^{-ik_1 r \cos(\varphi) - ik_2 r \sin(\varphi)} \\ &= \int_0^\infty dr \, \frac{1}{r} \int_{(2\pi)} d\varphi \, e^{i[2\varphi - kr \cos(\varphi - \alpha)]} \quad , \quad \varphi \rightarrow \varphi + \alpha - \pi/2 \\ &= -e^{i2\alpha} \int_0^\infty dr \, \frac{1}{r} \int_{(2\pi)} d\varphi \, e^{i[2\varphi - kr \sin(\varphi)]} \\ &= -e^{i2\alpha} \int_0^\infty dr \, \frac{1}{r} \Im \quad \text{with} \quad \Im = \int_{-\pi}^\pi d\varphi \, \cos(2\varphi - kr \sin(\varphi)) = 2\pi J_2(kr) \\ &= -e^{i2\alpha} \pi = -\frac{k_1 + ik_2}{k_1 - ik_2} \pi \quad , \end{aligned} \quad (13.21)$$

where for $\int_0^\infty dr \frac{1}{r} \Im = 2\pi \int_0^\infty dr \frac{1}{r} J_2(r) = \pi$ Abramovitz [20] was consulted under No. **11.4.16**, and for the Bessel function under No. **9.1.21**. Finally, to evaluate $\tilde{\kappa}$, the first steps can be read off from (13.21) (one $e^{i\varphi}$ less, one $1/r$ less) :

$$\begin{aligned} \tilde{\kappa}(\vec{k}) &= \int d^2r \frac{1}{z} e^{-i\vec{k}\vec{r}} = \int_0^\infty dr \int_{(2\pi)} d\varphi e^{i[\varphi - kr \cos(\varphi - \alpha)]} = -ie^{i\alpha} \int_0^\infty dr 2\pi J_1(kr) \\ &= \frac{k_1 + ik_2}{ik} \frac{2\pi}{k} = \frac{2\pi}{i} \frac{1}{k_1 - ik_2} . \end{aligned} \quad (13.22)$$

Herewith the details are completed for the translation of \mathbf{H} into \tilde{I} language. In the space of wave functionals $\psi[\tilde{I}^a(\vec{k})]$, the Hamiltonian reads

$$\begin{aligned} \mathbf{H}_{\text{Fourier}} &= m \int d^2k \tilde{I}^a(\vec{k}) \delta_{\tilde{I}^a(\vec{k})} - \pi m N \int d^2k k^2 \delta_{\tilde{I}^a(\vec{k})} \delta_{\tilde{I}^a(-\vec{k})} \\ &\quad - i m \int d^2q \int d^2p \frac{(p_1 - ip_2) \tilde{I}^c(\vec{p} + \vec{q})}{(q_1 - iq_2) + (p_1 - ip_2)} f^{cab} \delta_{\tilde{I}^a(\vec{q})} \delta_{\tilde{I}^b(\vec{p})} \\ &\quad + \frac{1}{4\pi m N} \int d^2k \tilde{I}^a(\vec{k}) \tilde{I}^a(-\vec{k}) . \end{aligned} \quad (13.23)$$

The suspected danger concerning singularities is banned. One can, of course, split off a factor $k = |\vec{k}|$ from $\tilde{I}^a(\vec{k})$. Thereby the factor k^2 in the second term is removed, but it appears in the last one, instead. The nonlinear f term has remained as terrible as before. The Fourier version (13.23) will develop some use in § 15.

14 More on the scalar product

If the Hamiltonian is understood as one half of the truth on the J quantum mechanics, then the exotic scalar product

$$\langle 1|2 \rangle = \int d\mu(\mathcal{C}) \psi_1^* \psi_2 = \sigma^n \int d\mu(\mathcal{H}) e^{2NS} \psi_1^* \psi_2 \equiv \sigma^n \int [dh^a] e^{2NS} \psi_1^* \psi_2 . \quad (14.1)$$

is the other half. The energy expectation value

$$\langle \mathbf{H} \rangle_\psi = \int d\mu(\mathcal{C}) \psi^* \mathbf{H} \psi \quad (14.2)$$

must be real. But how to show this? In § 14.4 it will turn out, indeed, that \mathbf{H} is “ $d\mu(\mathcal{C})$ –hermitean” (and that $d\mu(\mathcal{C})$ is real). Obviously, we might learn a bit more on making use of $\langle 1|2 \rangle$. For this, we must step down on the ladder $\varphi^a(\vec{r})$, $h^a(\vec{r})$, $h_{\text{KKN}}^a(\vec{r})$, $J^a(\vec{r})$, $\tilde{I}^a(\vec{k})$ of functional variables, where the “lowest” level is noticed at the left end, referring to $H = e^{\varphi^a T^a}$.

Stepping down, we remember four questions left open in part I. Question 1 concerns KKN’s differential variables dh_{KKN}^a (see (14.5) below), by which one leaves the space of

hermitean matrices H . This question will be answered immediately in the next subsection. Questions 2 to 4 are related to CFT. They will be bypassed by exploiting the Wilson loop argument — though, admittedly, arguing is not a very convincing behave. Question 2 was raised in the footnote below (9.1). It concerns the definite (?) expression σ as given by KKN :

$$[\mathbf{2.20}] : \quad \det(DD^\dagger) = \sigma^n e^{2NS} \quad \text{with} \quad \sigma \stackrel{?}{=} \frac{\det'(-\partial\bar{\partial})}{\int d^2r} \quad (14.3)$$

Perhaps, the problem merely is how to extract (14.3) from the material given in ref. 6 of KKN. Question 3 concerns the verification of

$$[\mathbf{2.26}] : \quad \sigma^n \int d\mu(\mathcal{H}) e^{2NS} \stackrel{?}{=} 1 \quad (14.4)$$

and could be an even less trivial problem of extraction, this time from [10, 9]. Help ! But note that σ is some real number, not involving functional variables. It could be absorbed into the measure $d\mu(\mathcal{H})$ or in ψ , as e.g. done in [9]. Missing σ anywhere, it is absorbed anyhow. Qestion 4 is the call for reasoning the fact that normalisable wave functionals depend on J^a , see (12.3) and comments below there. Now, here is our bypassing of the questions 2 to 4 by hand waving . If the Wilson loop argument (see also (16.21) and text around it) makes one to believe in the normalisability of functionals $\psi[J^a]$, namely by $\sigma^n \int d\mu(\mathcal{H}) e^{2NS} \psi^* \psi$, then we are allowed to set $\psi \equiv 1$, in particular, thus arriving at the l.h.s. of (14.4). As this l.h.s. is finite, (14.4) may be understood to define σ . But once σ is fixed this way, the question 14.3 is rather irrelevant.

14.1 The Haar measure was right

May it happen that, doing it wrong, the result is right? Sometimes. In § 8, around (8.7), we insisted in varying H by $H + \delta H = H^{1/2} (1 + dh^a \bar{T}^a) H^{1/2}$ since thereby staying within the space of hermitean matrices. Correspondingly, our integration measure turned out to be $d\mu(\mathcal{H}) = \prod_{\vec{r},a} dh^a =: [dh^a]$ with

$$dh^a = 2 \operatorname{Tr}(T^a H^{-1/2} dH H^{-1/2}) \quad . \quad dh_{\text{KKN}}^a = 2 \operatorname{Tr}(T^a H^{-1} dH) \quad , \quad (14.5)$$

on the other hand, is what KKN claim (below [2.15]) to be correct and what they call the *Haar measure for hermitean matrix-valued filelds*. The only possibility, for the two statements to differ in results, is a Jacobian $\neq 1$ of the change from variables h_{KKN}^a to h^a . The corresponding Jacobi matrix \mathfrak{S}^{ab} is obtained from

$$dh_{\text{KKN}}^a = \mathfrak{S}^{ab} dh^b \quad , \quad \text{i.e. from} \quad \operatorname{Tr}(T^a H^{-1} dH) = \mathfrak{S}^{ab} \operatorname{Tr}(T^b \rho^{-1} dH \rho^{-1}) \quad . \quad (14.6)$$

Remember that $H = \rho^2$. With concatenation in mind we may guess the solution :

$$\mathfrak{S}^{ab} = 2 \operatorname{Tr}(\rho T^a \rho^{-1} T^b) = 2 \operatorname{Tr}(T^b \rho T^a \rho^{-1}) =: \rho^{ba} = (\rho^T)^{ab} \quad . \quad (14.7)$$

To the right we recognise some adjoint version of ρ (cf. H^{ab} below (11.10)). Since $\det(\rho) = \det(\rho^T) = 1$ we get optimistic to have this property even adjointly. But how to show. One night, N. Dragon comes through the door and knows how :

$$\begin{aligned} \rho &= e^{\vec{\omega}\vec{T}} \quad , \quad \rho T^a \rho^{-1} = T^a + \left[\vec{\omega}\vec{T}, T^a \right] + \dots = \dots = (e^{i\omega^c f^c \dots})^{ab} T^b \\ \mathfrak{S}^{ab} &= 2 \operatorname{Tr}((e^{\dots})^{ad} T^d T^b) = (e^{\dots})^{ab} \\ \det(\mathfrak{S}) &= e^{\operatorname{Sp}(\ln(e^{\dots}))} = e^{i\omega^c f^{cdd}} = e^0 = 1 \quad \text{and hence} \\ &\int [dh^a] \dots = \int [dh_{\text{KKN}}^a] \dots \quad . \end{aligned} \quad (14.8)$$

Nice, all are right. May be, it can be seen more directly. Just looking at (14.5), M. Flohr realised it to be a change to new generators $\rho^{-1} T^a \rho$. Now obtaining a unit Jacobian, he said, one could expect.

14.2 Variables φ^a

A pedestrian (what does it mean?) likes the elementary tools in favour of stability and explicitity, and he has all time of the world for going ahead. Clearly, he favours the “lowest” type φ^a of functional variables. They are defined by $H = e^{\vec{\varphi}\vec{T}}$, hence real, found already in [2.15] and related to our former parameters ω^a ($\rho = e^{\vec{\omega}\vec{T}}$) by $\varphi^a = 2\omega^a$, simply. The linear relation between $d\varphi^a$ and dh^a defines a Jacobian matrix. But dh^a comes in two versions, giving two linear relations, namely

$$\text{OURS} \quad dh^a = s^{ab} d\varphi^b \quad \text{and KKN's} \quad dh_{\text{KKN}}^a = d\varphi^b r^{ba} \quad . \quad (14.9)$$

Using (14.5) and (8.14) the matrices s and r become explicit :

$$s^{ab} = \int_{-1}^1 ds \operatorname{Tr} \left(T^a e^{s\vec{\omega}\vec{T}} T^b e^{-s\vec{\omega}\vec{T}} \right) \quad , \quad r^{ab} = \int_0^1 ds \, 2 \operatorname{Tr} \left(T^a e^{s\vec{\varphi}\vec{T}} T^b e^{-s\vec{\varphi}\vec{T}} \right) \quad . \quad (14.10)$$

Due to (14.6), i.e. $dh_{\text{KKN}}^a = \mathfrak{S}^{ab} dh^b$, and $\mathfrak{S}^{ab} = \rho^{ba}$, they are related by

$$r^{ab} = \rho^{cb} s^{ca} = s^{ac} \rho^{cb} \quad , \quad (14.11)$$

because $s^{ab} = s^{ba}$. Perhaps (14.11) is worth to be checked (it works so well) :

$$\begin{aligned} r^{ab} &= \int_0^1 ds \, 2 \operatorname{Tr} \left(T^a e^{(2s-1)\vec{\omega}\vec{T}} e^{\vec{\omega}\vec{T}} T^b e^{-\vec{\omega}\vec{T}} e^{(1-2s)\vec{\omega}\vec{T}} \right) \quad , \quad 2s-1 = s' \\ &= \int_{-1}^1 ds' \operatorname{Tr} \left(e^{-s'\vec{\omega}\vec{T}} T^a e^{s'\vec{\omega}\vec{T}} e^{\vec{\omega}\vec{T}} T^b e^{-\vec{\omega}\vec{T}} \right) \\ &= \int_{-1}^1 ds' \operatorname{Tr} \left(e^{-s'\vec{\omega}\vec{T}} T^a e^{s'\vec{\omega}\vec{T}} T^c \right) 2 \operatorname{Tr} (T^c \rho T^b \rho^{-1}) \quad , \quad \text{q.e.d.} \end{aligned} \quad (14.12)$$

Organized a bit different the above calculation would have led to $r^{ab} = \rho^{ac} s^{cb}$: s and ρ do commute, indeed. The matrix s^{ab} is symmetric and real, while r and ρ are only hermitean. But note that

$$\det(r) = \det(s) = \text{real} \quad \curvearrowright \quad d\mu(\mathcal{C}) = \text{real} \quad . \quad (14.13)$$

Admittedly, r is the more relevant matrix, as is seen shortly.

14.3 Hermitean adjoint with respect to $d\mu(\mathcal{H})$

Some sense behind the above stepping back to the “lowest” variable becomes obvious, when now asking for the meaning of a dagger in our functional quantum mechanics — dagger with respect to which integration measure? Given a functional operator Q , then let its adjoint operator Q^\dagger with respect to $d\mu(\mathcal{H}) = [dh^a]$ be defined by

$$\int d\mu(\mathcal{H}) (Q\chi_1)^* \chi_2 =: \int d\mu(\mathcal{H}) \chi_1^* Q^\dagger \chi_2 \quad . \quad (14.14)$$

To determine Q^\dagger , the relation $[dh^a] = [d\varphi^a] \det(r)$ will help doing partial integrations with δ_{φ^a} . It looks bad. So, we might be satisfied with just one example for an operator Q . Aside, we now enter KKN’s § 3.3 : “ p, \bar{p} as adjoints for $d\mu(H)$ ”.

Let the given (n -fold) operator Q be

$$p^a := -i(r^{-1})^{ab} \delta_{\varphi^b} = \partial(M^{-1})^{ac} \delta^c = (11.7) \quad . \quad (14.15)$$

So far, the functional operator p^a , merely played a minor role in § 11.3. Here it is revalued. To derive the left expression in (14.15) from the known right one, show at first that

$$p_r^a H_{r'} = -i H T^a \delta(\vec{r} - \vec{r}') \quad (14.16)$$

by using $\delta^c M_{r'}^\dagger = 0$ and $\delta^c M_{r'}$ as given in the text below (13.4). The commutator version of (14.16) is [2.39]. Now, using the ansatz $p^a = X^{ab} \delta_{\varphi^b}$ (as well as $H = e^{\vec{\varphi} \vec{T}}$), one obtains the matrix X as $X^{ab} = -i(r^{-1})^{ab}$.

The with-respect-to- $d\mu(\mathcal{H})$ adjoint operator of p^a , as claimed by KKN in [3.12], is $\bar{p}^a := -p^{a*} = -i(r^{*-1})^{ab} \delta_{\varphi^b}$. We verify⁵ :

$$\begin{aligned} \int [dh^a] (p^a \chi_1)^* \chi_2 &= \int [d\varphi^a] \det(r) (-i(r^{-1})^{ab} \delta_{\varphi^b} \chi_1)^* \chi_2 \\ &= \int [d\varphi^a] \det(r) i (r^{*-1})^{ab} \chi_2 \delta_{\varphi^b} \chi_1^* = \int [d\varphi^a] \chi_1^* \delta_{\varphi^b} \det(r) (-i) (r^{*-1})^{ab} \chi_2 \\ &= \int [d\varphi^a] \det(r) \chi_1^* (-i) (r^{*-1})^{ab} \delta_{\varphi^b} \chi_2 - i \int [d\varphi^a] \chi_1^* \chi_2 \delta_{\varphi^b} \det(r) (r^{*-1})^{ab} \\ &= \int [dh^a] \chi_1^* \bar{p}^a \chi_2 - i \int [d\varphi^a] \chi_1^* \mathcal{O}^{a*} \chi_2 \quad \text{with} \end{aligned} \quad (14.17)$$

$$\mathcal{O}^a = \delta_{\varphi^b} \det(r) (r^{-1})^{ab} = \lim_{\vec{r}' \rightarrow \vec{r}} \mathcal{P}^a \quad , \quad \mathcal{P}^a = \delta_{\varphi_{r'}^b} \det(r) (r^{-1})^{ab} \quad . \quad (14.18)$$

If KKN’s statement is right, \mathcal{O}^a must vanish. We question, whether the limit in (14.18) needs care towards regularization (as e.g. KKN take), or whether perhaps the following quick way is sufficient, which obtains $\mathcal{P}^a = 0$ even before the limit $\vec{r}' \rightarrow \vec{r}$ is performed.

⁵There is an infinity of $\det(r)$ ’s in the functional integrations. In (14.17), only the one is made explicit, which carries the same spatial index as δ_{φ^b}

Let's see. Could-be dangerous steps carry a $\not\Leftarrow$ mark :

$$\begin{aligned}
\mathcal{P}^a &= (r^{-1})^{ab} \delta_{\varphi_{r'}}^b \det(r) + \det(r) \delta_{\varphi_{r'}}^b (r^{-1})^{ab} \\
&\stackrel{\not\Leftarrow}{=} \det(r) \left[\underbrace{(r^{-1})^{ab} \delta_{\varphi_{r'}}^b \ln(\det(r))}_{= \delta_{\varphi_{r'}}^b \text{Tr}(\ln(r))} + \delta_{\varphi_{r'}}^b (r^{-1})^{ab} \right] \\
&= \det(r) (r^{-1})^{ad} (r^{-1})^{cb} \left\{ \delta_{\varphi_{r'}}^d r^{bc} - \delta_{\varphi_{r'}}^b r^{dc} \right\} \quad \not\Leftarrow\Leftarrow \\
&= \det(r) (r^{-1})^{ad} (r^{-1})^{cb} \left\{ \delta_{\varphi_{r'}}^d r^{bc} - \delta_{\varphi_{r'}}^b r^{dc} \right\} \quad . \quad (14.19)
\end{aligned}$$

Given this intermediate result, the desired number zero is derived without further danger :

$$\begin{aligned}
\delta_{\varphi_{r'}}^d r^{bc} &= \delta_{\varphi_{r'}}^d \int_0^1 ds \, 2 \text{Tr} (b e^s c e^{-s}) \quad \begin{array}{l} \text{the short-hand notation} \\ \text{might be obvious from (14.10)} \end{array} \\
&= \int_0^1 ds \int_0^1 dt \, s \, 2 \text{Tr} (c e^{-s} b e^{ts} d e^{(1-t)s} - b e^s c e^{-ts} d e^{-(1-t)s}) \delta(\vec{r}' - \vec{r}) \\
&= \delta(\vec{r}' - \vec{r}) \int_0^1 ds \int_0^s dt \, 2 \text{Tr} (c e^{-s} b e^t d e^{s-t} - b e^s c e^{-t} d e^{t-s}) \\
&= \delta(\vec{r}' - \vec{r}) \int_0^1 ds \int_0^s dt \, 2 \text{Tr} (c e^{-s} b e^s e^{-t} d e^t - e^{-t} d e^t e^{-s} b e^s c) \quad , \quad (14.20)
\end{aligned}$$

where the first term in the last line was obtained through $t \rightarrow s - t$, leaving the borders of t -integration unchanged. The other expression in the wavy bracket of (14.19), namely $-\delta_{\varphi_{r'}}^b r^{dc}$, equals minus the first one through $b \hookrightarrow d$. It can be further converted by $\int_0^1 ds \int_0^s dt = \int_0^1 dt \int_t^1 ds$ and $s \hookrightarrow t$, finally :

$$\begin{aligned}
-\delta_{\varphi_{r'}}^b r^{dc} &= \delta(\vec{r}' - \vec{r}) \int_0^1 ds \int_0^s dt \, 2 \text{Tr} (e^{-t} b e^t e^{-s} d e^s c - c e^{-s} d e^s e^{-t} b e^t) \\
&= \delta(\vec{r}' - \vec{r}) \int_0^1 ds \int_s^1 dt \, 2 \text{Tr} (c e^{-s} b e^s e^{-t} d e^t - e^{-t} d e^t e^{-s} b e^s c) \quad . \quad (14.21)
\end{aligned}$$

Constituting the wavy bracket $\left\{ \delta_{\varphi_{r'}}^d r^{bc} - \delta_{\varphi_{r'}}^b r^{dc} \right\}$, two pieces of the t integral fit together :

$$\begin{aligned}
\{ \quad \} &= \delta(\vec{r}' - \vec{r}) \int_0^1 ds \int_0^1 dt \, 2 \text{Tr} ([c e^{-s} b e^s - e^{-s} b e^s c] e^{-t} d e^t) \\
&= \delta(\vec{r}' - \vec{r}) \int_0^1 ds \int_0^1 dt \, 2 \text{Tr} (e^{-s} b e^s [h, c]) \, 2 \text{Tr} (h e^{-t} d e^t) \quad , \quad [c, h] = i f^{hcj} j \\
&= \delta(\vec{r}' - \vec{r}) i f^{hcj} r^{bj} r^{dh} \quad . \quad (14.22)
\end{aligned}$$

Inserting this into (14.19) we arrive at

$$\mathcal{P}^a = \det(r) \delta(\vec{r}' - \vec{r}) i f^{hcj} \delta^{ah} \delta^{cj} \sim f^{acc} = 0 \quad , \quad (14.23)$$

as announced. Never perform $t \rightarrow s - t$ in the wrong term or de-concatenation with an unfortunate content of trace. Admittedly, we could not understand one detail in KKN, namely to get [3.14] obvious immediately. But it is correct, this damned equation. To summarize,

$$\int [dh^a] \chi_1^* \bar{p}^a \chi_2 = \int [dh^a] (p^a \chi_1)^* \chi_2 \quad . \quad (14.24)$$

14.4 \mathbf{H} is self-adjoint with respect to $d\mu(\mathcal{C})$

What else?! The headline must be valid, because all the former treatment was sound and stable. Hence, $\langle \mathbf{H} \rangle$, (14.2), is real. End. However, to verify the headline, is also a very crucial test. So, for confidence, let us show that

$$\langle 1 | \mathbf{H} | 2 \rangle = \int d\mu(\mathcal{C}) \psi_1^* \mathbf{H} \psi_2 \stackrel{?!}{=} \int d\mu(\mathcal{C}) (\mathbf{H} \psi_1)^* \psi_2 = \langle \mathbf{H} | 1 | 2 \rangle \quad . \quad (14.25)$$

The potential term \mathbf{V} , being real and containing no functional derivatives, drops out in (14.25) immediately. For \mathbf{T} the “lower” version $\mathbf{T} = \int \frac{e^2}{2} H^{ab} (\overline{Gp})^a (Gp)^b$ of (11.10) is convenient by two reasons: (1) we know about the p^a operators how they become adjoint with respect to the measure $d\mu(\mathcal{H})$, and (2) we know from (11.16) how $(Gp)^a$ is regularized, namely as $\int' \mathcal{G}_{rr'}^{ab} p_{r'}^b$. By the way, this was the reason for always writing already $(Gp)^a := \int' G_{rr'} p_{r'}^a$ with an a index outside of parantheses. We go our own way :

$$\begin{aligned} \int d\mu(\mathcal{C}) \psi_1^* \mathbf{T} \psi_2 &= \frac{e^2}{2} \int_r \sigma^n \int [dh^a] e^{2NS} \psi_1^* H^{ab} (\overline{Gp})^a (Gp)^b \psi_2 \quad , \quad H^{ab*} = H^{ba} \quad , \quad \overline{G} = G^* \\ &= \frac{e^2}{2} \int_r \sigma^n \int [dh^a] \left[(Gp)^a e^{2NS} H^{ba} \psi_1 \right]^* (Gp)^b \psi_2 \\ &= \frac{e^2}{2} \int_r \sigma^n \int [dh^a] \left[(\overline{Gp})^b (Gp)^a e^{2NS} H^{ba} \psi_1 \right]^* \psi_2 \\ &= \frac{e^2}{2} \int_r \int d\mu(\mathcal{C}) \left[e^{-2NS} (\overline{Gp})^a (Gp)^b e^{2NS} H^{ab} \psi_1 \right]^* \psi_2 \quad . \end{aligned} \quad (14.26)$$

It is seen what we need. The whole apparatus $e^{2NS} H^{ab}$ might commute with the $(Gp)(Gp)$ block. So, the first interesting commutator is

$$\begin{aligned} \left[(Gp)^b, e^{2NS} H^{ab} \right] &= \left[(Gp)^b, e^{2NS} \right] H^{ab} + e^{2NS} \left[(Gp)^b, H^{ab} \right] \\ &= e^{2NS} \left\{ 2N H^{ab} (Gp)^b S + (Gp)^b H^{ab} \right\} \stackrel{?}{=} 0 \quad . \end{aligned} \quad (14.27)$$

Obviously, the two terms in the wavy bracket might compensate. Show!

First term. Our knees turn to jelly. S , isn't that this awkward WZW action with all its volume term complication? How to avoid doing explicit calculations with? The neat aspect of the action S was its functional differential equation (9.35), i.e. $\delta S = \frac{1}{\pi} \int \text{Tr} (H^{-1} (\delta H) \partial H^{-1} \overline{\partial} H)$ with ∂ acting on all factors to the right of. Indeed, δS is all what we need for performing the (Gp) derivative. When performing $(Gp)^b S$, (Gp) may remain unregularized, with the reason for that already the S differential equations had been regularized :

$$\begin{aligned} 2N H^{ab} (Gp)^b S &= 2N H_r^{ab} \int' G_{rr'} (-i r_{r'}^{-1})^{bc} \frac{1}{\pi} \int'' \text{Tr} \left(H_{r''}^{-1} \left[\delta_{\varphi_{r'}}^b H_{r''} \right] \partial'' H_{r''}^{-1} \overline{\partial}'' H_{r''} \right) \\ &= 2N H_r^{ab} \int' G_{rr'} \frac{1}{\pi} \int'' \text{Tr} \left(H_{r''}^{-1} \left[p_{r'}^b H_{r''} \right] \partial'' H_{r''}^{-1} \overline{\partial}'' H_{r''} \right) \\ (14.16) : \quad &= -i 2N H_r^{ab} \int' G_{rr'} \frac{1}{\pi} \text{Tr} \left(T^b \partial' H_{r'}^{-1} \overline{\partial}' H_{r'} \right) \end{aligned}$$

$$\begin{aligned}
&= i 2N H_r^{ab} \int' \frac{1}{\pi} \text{Tr} \left(T^b H_r^{-1} \bar{\partial}' H_r \right) \partial' G_{rr'} \\
&= -i \frac{N}{\pi} 2 \text{Tr} \left(T^a H T^b H^{-1} \right) 2 \text{Tr} \left(T^b H^{-1} \bar{\partial} H \right) \\
&= -i \frac{N}{\pi} 2 \text{Tr} \left(T^a (\bar{\partial} H) H^{-1} \right) \quad . \tag{14.28}
\end{aligned}$$

Second term. One can foresee a spatial delta function from differentiating H , hence a coincidence limit in the Greens function. So, the finite value (11.27) of \mathcal{G}_{rr} will have its use again :

$$\begin{aligned}
(Gp)_{\text{reg}}^b H^{ab} &= \int' \mathcal{G}_{rr'}^{bc} p_{r'}^c 2 \text{Tr} \left(T^a H T^b H^{-1} \right) \\
&= \int' \mathcal{G}_{rr'}^{bc} 2 \text{Tr} \left(T^a [p_{r'}^c H] T^b H^{-1} - T^a H T^b H^{-1} [p_{r'}^c H] H^{-1} \right) \\
(14.16) : &= -i \mathcal{G}_{rr}^{bc} 2 \text{Tr} \left(T^a H T^c T^b H^{-1} - T^a H T^b T^c H^{-1} \right) = f^{cbd} \mathcal{G}_{rr}^{bc} H^{ad} \\
(11.27) : &= \frac{1}{\pi} H^{ad} f^{cbd} \left[H^{-1} \bar{\partial} H \right]^{bc} = \frac{1}{\pi} H^{ad} f^{cbd} H^{eb} \bar{\partial} H^{ec} \\
&= \frac{-i}{\pi} 2 \text{Tr} \left(H^{-1} a H [c, b] \right) 2 \text{Tr} \left(b H^{-1} e H \right) 2 \text{Tr} \left(c \bar{\partial} H^{-1} e H \right) \\
&= \frac{-i}{\pi} 2 \text{Tr} \left(H^{-1} [e, a] H c \right) 2 \text{Tr} \left(c \bar{\partial} H^{-1} e H \right) \\
&= \frac{-i}{\pi} 2 \text{Tr} \left(H^{-1} [[e, a], e] \bar{\partial} H \right) \\
&= + i \frac{N}{\pi} 2 \text{Tr} \left(T^a (\bar{\partial} H) H^{-1} \right) = \text{minus (14.28)} \quad . \tag{14.29}
\end{aligned}$$

Thus, the vanishing of (14.27) is obtained, indeed, and $(Gp)^b$ commutes with $e^{2NS} H^{ab}$. Next we need a vanishing commutator also with $(\bar{G}\bar{p})^a$. But this, after all, is easily shown by taking (14.27) conjugate complex and at interchanged indices a, b . The long chain of equations (14.26) now comes to its end :

$$\int d\mu(\mathcal{C}) \psi_1^* \mathbf{T} \psi_2 = \frac{\epsilon^2}{2} \int_r \int d\mu(\mathcal{C}) \left[H^{ab} (\bar{G}\bar{p})^a (Gp)^b \psi_1 \right]^* \psi_2 = \int d\mu(\mathcal{C}) (\mathbf{T} \psi_1)^* \psi_2 \quad . \tag{14.30}$$

Quod erat demonstrandum.

Presumably, that statement in [5] below **{ 17 }**, “*In principle, one may also obtain the measure of integration for the inner product of the wavefunctions by requiring self-adjointness of the above expression*” may be understood as follows. In order to get \mathbf{T} self-adjoint with respect to $d\mu(\mathcal{C})$, (14.28) must hold true, in particular its first line, i.e. the functional differential equations for S . But having these, one can even turn to its WZW solution.

15 Vacuum and first excitations

The energy spectrum of the 2+1D Yang–Mills system follows from its functional stationary Schrödinger equation

$$(\mathbf{T} + \mathbf{V})\psi = E\psi \quad , \quad (15.1)$$

where we look to e.g. the framed J version (13.14) of the Hamiltonian. Every \mathbf{T} term exhibits a functional derivative at its right end. Were there no part \mathbf{V} , we had one solution immediately, namely $\psi_{00} \equiv 1$ with eigenvalue $E = 0$ (let the second index refer to the approximation $V \approx 0$). Hence, we know the vacuum state in this approximation. According to (14.4) it is even normalized right. But \mathbf{V} is not zero.

15.1 The vacuum wave function ψ_0

There are several possibilities of extracting one or the other truth from (15.1). But, most probably, this equation cannot be solved exactly. In the spirit of “ \mathbf{V} is small”, in a sense to be developed, KKN propose a systematic iterative procedure (**{ 19 }** to **{ 24 }**), which results in an expansion of the vacuum wave function ψ_0 in powers of $1/m^2$. Its zeroth approximation is $\psi_{00} \equiv 1$. It is field theory. So, let the vacuum continue to have energy zero: $(\mathbf{T} + \mathbf{V})\psi_0 = 0$. We follow KKN and seek ψ_0 in the form

$$\psi_0 [J] = e^{P[J]} \psi_{00} = e^P 1 \quad . \quad (15.2)$$

Inserting this into (15.1) (and taking $E = 0$ there),

$$e^{-P} (\mathbf{T} + \mathbf{V}) e^P \psi_{00} = 0 \quad , \quad \text{i.e.} \quad \tilde{H} 1 = 0 \quad \text{with} \quad \tilde{H} := e^{-P} (\mathbf{T} + \mathbf{V}) e^P \quad , \quad (15.3)$$

we enjoy the fact that the series of commutators,

$$\tilde{H} = e^{-P} (\mathbf{T} + \mathbf{V}) e^P = \mathbf{T} + [\mathbf{T}, P] + \frac{1}{2} \left[[\mathbf{T}, P], P \right] + \mathbf{V} \quad , \quad (15.4)$$

breaks off, because \mathbf{T} contains two functional derivatives, at highest. The last two terms in (15.4) do no more differentiate. From the condition $\tilde{H} 1 = 0$, i.e.,

$$0 \stackrel{!}{=} \tilde{H} 1 = [\mathbf{T}, P] 1 + \frac{1}{2} \left[[\mathbf{T}, P], P \right] + \mathbf{V} \quad , \quad (15.5)$$

the functional $P = P_1 + P_2 + P_3 + \dots$ can now be determined term by term. Due to $\mathbf{T} \sim m$, $\mathbf{V} \sim m^{-1}$, and $P_j \sim m^{-2j}$ ($j = 1, 2, \dots$) we have

$$\begin{aligned} m^{-1} : \quad 0 &= [\mathbf{T}, P_1] 1 + \mathbf{V} && \leadsto P_1 \\ m^{-3} : \quad 0 &= [\mathbf{T}, P_2] 1 + \frac{1}{2} \left[[\mathbf{T}, P_1], P_1 \right] && \leadsto P_2 \\ m^{-5} : \quad 0 &= [\mathbf{T}, P_3] 1 + \frac{1}{2} \left[[\mathbf{T}, P_2], P_1 \right] + \frac{1}{2} \left[[\mathbf{T}, P_1], P_2 \right] && \leadsto P_3 \end{aligned} \quad (15.6)$$

and so on. In **{ 23 }** KKN state the result for P_1 , P_2 and P_3 . But here, we shall be satisfied with P_1 and rather study the delicate normal ordering. Also, we switch to the Fourier version (13.23). P_1 cannot contain a third power of I , because $\mathbf{T}_1 \sim I \delta$ reproduces the I power and otherwise there is only $\mathbf{V} \sim I^2$ in (15.6). An additive constant in P_1 would drop out in (15.6). Hence, we set

$$P_1 = \alpha \int d^2 q \tilde{I}^a(\vec{q}) \tilde{I}^a(-\vec{q}) \quad . \quad (15.7)$$

(15.6) has no contribution from $\mathbf{T}_3 \sim I^c f^{cab} \delta^a \delta^b$, because via $[\delta \delta, I^2] = \delta [\delta, I^2] + [\delta, I^2] \delta = 2 + 4I\delta$ the second term $4I\delta$ vanishes when applied to 1, and “2” actually is $2\delta^{ab}$ and gives $f^{caa} = 0$. There remains

$$\begin{aligned} 0 &= \alpha m \int d^2 k \tilde{I}^a(\vec{k}) \left[\delta_{\tilde{I}^a(\vec{k})}, \int d^2 q \tilde{I}^a(\vec{q}) \tilde{I}^a(-\vec{q}) \right] 1 \\ &\quad - \alpha \pi m N \int d^2 k k^2 \left[\delta_{\tilde{I}^a(\vec{k})} \delta_{\tilde{I}^a(-\vec{k})}, \int d^2 q \tilde{I}^a(\vec{q}) \tilde{I}^a(-\vec{q}) \right] 1 \\ &\quad + \frac{1}{4\pi m N} \int d^2 k \tilde{I}^a(\vec{k}) \tilde{I}^a(-\vec{k}) \\ &= 2\alpha m \int d^2 k \tilde{I}^a(\vec{k}) \tilde{I}^a(-\vec{k}) - \alpha \pi m N \int d^2 k k^2 2 \delta_{\tilde{I}^a(\vec{k})} \tilde{I}^a(\vec{k}) \\ &\quad + \frac{1}{4\pi m N} \int d^2 k \tilde{I}^a(\vec{k}) \tilde{I}^a(-\vec{k}) \end{aligned} \quad (15.8)$$

with an obvious catastrophe in the overlaid term. It derived from the second line through $[\delta \delta, I^2] 1 = \delta [\delta, I^2] 1 = \text{“2”}$. $\delta_{\tilde{I}^a(\vec{k})} \tilde{I}^a(\vec{k}) = n \delta(0)$ – ? – this is unthinkable. By including \mathbf{V} we runned into a typical field theoretical infinity. If the groundstate energy was to be kept at $E = 0$ then \mathbf{V} had to be normal ordered from the outset. Concentrating to the I^2 terms of (15.8), one is led to $\alpha = -1/(8\pi m^2 N)$ and

$$P_1 = -\frac{1}{8\pi m^2 N} \int d^2 q \tilde{I}^a(\vec{q}) \tilde{I}^a(-\vec{q}) \quad . \quad (15.9)$$

To maintain this conclusion, one misses one term in the last line of (15.8), namely $-\frac{1}{8m} \int d^2 k k^2 2 \delta_{\tilde{I}^a(\vec{k})} \tilde{I}^a(\vec{k})$. By including this missing term, one turns from \mathbf{V} to $:\mathbf{V}:$. In the next subsection we make an attempt to understand this a bit better.

No problems arise in transporting the result (15.9) from its Fourier underworld back to the J version. By means of (13.16) this looks as

$$\begin{aligned} P_1 &= -\frac{(2\pi)^2}{8\pi m^2 N} \int d^2 r I^a(\vec{r}) I^a(\vec{r}) \\ &= -\frac{\pi}{2m^2 N} \int (\bar{\partial} J^a) \bar{\partial} J^a = -\frac{\pi}{m^2 N} \text{Tr} \left(\int (\bar{\partial} J) \bar{\partial} J \right) \quad . \end{aligned} \quad (15.10)$$

In the last step concatenation was used and the definition $J^a =: 2 \text{Tr}(T^a J)$. There is no perfect agreement of our last expression in (15.10) with **{ 23 }** (first term), because KKN have normal ordering colons around it.

To summarize, to its leading order, the vacuum wave functional is obtained as

$$\begin{aligned}\psi_0 &= e^P 1 = e^{P_1 + O(m^{-4})} \approx e^{P_1} = e^{-\frac{\pi}{2m^2 N} \int (\bar{\partial} J^a) \bar{\partial} J^a} \\ &= e^{-\frac{1}{2m} \mathbf{V}} = e^{-\frac{1}{2m\epsilon^2} \int \text{Sp}(BB)} \quad .\end{aligned}\tag{15.11}$$

In the second line (less important) **{ 24 }** is reached, and just (13.10) was inserted for \mathbf{V} . But the underlined version in the first line will be basic to the derivation of confinement in § 17.

15.2 Analogy with a 1D oscillator

From the viewpoint of a poor man, familiar with non-relativistic quantum mechanics, the above was a somewhat strange procedure. Assume, someone works with the norm $\langle 1|2 \rangle = \pi^{-1/2} \int dx e^{-x^2} \varphi_1^* \varphi_2$ and would like to solve

$$H\varphi = E\varphi \quad \text{with} \quad H = mx\partial_x - \frac{m}{2}\partial_x^2 + \frac{k^2}{2m}x^2 =: T + V \quad .\tag{15.12}$$

Obviously, this was inspired by (13.23) (with $\tilde{I} = k\sqrt{2\pi N}x$ in mind). $\varphi \equiv 1$ is correctly normalized, but only for $V = 0$ it is a solution to the eigenvalue equation.

For solving exactly, he turns back to the usual norm by $\varphi = \pi^{1/4} e^{x^2/2} \vartheta$, hence to the Hamiltonian $\bar{H} = e^{-x^2/2} H e^{x^2/2}$. Then, through $e^{-x^2/2} \partial_x e^{x^2/2} = \partial_x + x$, he obtains

$$\begin{aligned}\bar{H}\vartheta &= E\vartheta \quad , \quad \bar{H} = -\frac{m}{2}\partial_x^2 + \frac{m}{2} \left[1 + \frac{k^2}{m^2} \right] x^2 - \frac{m}{2} \quad . \quad n = 0, 1, 2, \dots : \\ E_n &= \frac{1}{2} \left(-m + \sqrt{m^2 + k^2} \right) + n \sqrt{m^2 + k^2} \quad , \quad \vartheta_0 = e^{-\frac{1}{2}\sqrt{1+k^2/m^2} x^2} \quad ,\end{aligned}\tag{15.13}$$

because he recognized an “oscillator with $\hbar = m$ and $\omega = \sqrt{1 + k^2/m^2}$ ”. If $k^2 = 0$ one would define the annihilators $\bar{b} = (x + \partial_x)/\sqrt{2}$ and creators $b^\dagger = (x - \partial_x)/\sqrt{2}$. Herewith, and using $b = e^{x^2/2} \bar{b} e^{-x^2/2}$ etc., we now turn back to the original space, obtaining

$$\varphi_0 = e^{\frac{1}{2}(1 - \sqrt{1+k^2/m^2})x^2} = e^{-\frac{k^2}{4m^2}x^2 + O(k^4)} \quad , \quad b = \frac{1}{\sqrt{2}}\partial_x \quad , \quad b^\dagger = \frac{1}{\sqrt{2}}(2x - \partial_x) \quad .\tag{15.14}$$

Note that, by analogous simplifications, P_1 from (15.9) turns into $-k^2 x^2/(4m^2)$ and agrees with the above exponent of φ_0 . The two kinetic terms of (15.12) are $mb^\dagger b$, so normal ordering has no effect. But in V it has, because due to $x = (b + b^\dagger)/\sqrt{2}$ it is

$$x^2 = \frac{1}{2} (b^2 + b^{\dagger 2} + bb^\dagger + b^\dagger b) \quad , \quad :x^2: = \frac{1}{2} (b^2 + b^{\dagger 2} + 2b^\dagger b) = \dots = x^2 - \frac{1}{2} \quad .\tag{15.15}$$

Let us now treat the oscillator with KKN’s P method, however using $:V:$ in (15.12) from the outset. Then, the ansatz $P_1 = \alpha x^2$ (no matter whether normal ordered or not) leads to the (15.8) analogue

$$0 = m (2x^2 - 1) \left(\alpha + \frac{k^2}{4m^2} \right)\tag{15.16}$$

and to the lowest-order groundstate function φ_0 of (15.14), indeed. Applying $:H:$ to this function, the eigenvalue $E = 0$ might be obtained to lowest order in k^2 . This is indeed the case :

$$:H: e^{-\frac{k^2}{4m^2}x^2} = \left\{ -\frac{k^2}{2m}x^2 - \frac{m}{2} \left[\frac{k^4 x^2}{4m^4} - \frac{k^2}{2m^2} \right] + \frac{k^2}{2m} \left(x^2 - \frac{1}{2} \right) \right\} e^{-\frac{k^2}{4m^2}x^2} = 0 \quad (15.17)$$

We do not claim that herewith the § 15.1 is fully understood.

15.3 $\sqrt{m^2 + k^2}$

By turning to the excited states, we shall run into certain difficulties. So, let this subsection 15.3 be tentative and end up with open questions, see the remarks at its end.

In search of a starting point, there are two possible brute-force simplifications of the Hamiltonian (13.23) : [1.] $\mathbf{V} = 0$ and [2.] $\text{NT} = 0$, where NT stands for "nonlinear term", i.e. for the f^{cab} term in (13.23).

[1.] $\mathbf{V} = 0$: hence $\mathbf{H} = \mathbf{T}$, and we have $\mathbf{T}1 = 0$, and $\mathbf{T}\tilde{I}^a(\vec{k}) = m\tilde{I}^a(\vec{k})$, while in higher excited states the NT is involved, perhaps forming glue ball states with increasing content. We are interested in the first excited state and the fate of its infinite degeneracy under inclusion of \mathbf{V} . Including \mathbf{V} to its first order leads to the modification $m \rightarrow m + k^2/(2\pi)$ of the eigenvalue (we jump over the derivation, because this will be seen shortly to be contained in the following at small- k). But to second order in \mathbf{V} the NT comes into play, and we run in essence into the same difficulties as [2.] So, [1.] is a poor starting point.

The first step towards the inclusion of \mathbf{V} or NT, or both, is exact. In general, for solving the stationary Schrödinger equation $\mathbf{H}\psi = E\psi$, one can write it as

$$e^{-P}\mathbf{H}e^P (e^{-P}\psi) = \tilde{H} (e^{-P}\psi) = E (e^{-P}\psi) \quad (15.18)$$

with an arbitrary function(al) P . In particular, we are allowed to fix P by $\tilde{H}1 = 0$, as in § 15.1. Assume that P has been determined exactly this way, then the V term can be eliminated from the Hamiltonian — by subtracting the number zero (15.5) :

$$\begin{aligned} \tilde{H} &= \mathbf{T} + [\mathbf{T}, P] + \frac{1}{2} \left[[\mathbf{T}, P], P \right] + \mathbf{V} \\ &\quad - [\mathbf{T}, P]1 - \frac{1}{2} \left[[\mathbf{T}, P], P \right] - \mathbf{V} \\ &= \mathbf{T} + \left\{ [\mathbf{T}, P] - [\mathbf{T}, P]1 \right\} \quad (15.19) \end{aligned}$$

The last term just cancels terms of $[\mathbf{T}, P]$, which do not differentiate. Hence $\tilde{H}1 = 0$ is fulfilled, indeed. The potential is now hidden in P through (15.5). Using the exact P , all powers of I will appear in front of the first differentiation. Symbolically, the Schrödinger equation reads

$$\tilde{H}\chi = \left[(I + I^2 + I^3 + \dots) \delta + \delta\delta + If\delta\delta \right] \chi = E\chi \quad (15.20)$$

Already the inclusion I^2 in the round bracket (but no more I^3), requires χ to contain all powers of I .

[2.] Omitting the NT, the Hamiltonian becomes a quadratic form in I and δ . Without NT, each contribution P_1, P_2, P_3 etc. reduces to its term quadratic in the field. Instead of summing them up, it is quite easier, of course, to solve (15.5) directly with the ansatz

$$P_{\text{qu}} = \int d^2 q \tilde{I}^a(\vec{q}) \alpha(q^2) \tilde{I}^a(-\vec{q}) \quad . \quad (15.21)$$

for the function α . Using P_{qu} in (15.19) we have

$$\begin{aligned} [\mathbf{T}, P_{\text{qu}}] &= m \int d^2 k \tilde{I}^a(\vec{k}) \left[\delta_{\tilde{I}^a(\vec{k})}, P_{\text{qu}} \right] - \pi m N \int d^2 k k^2 \left[\delta_{\tilde{I}^a(\vec{k})} \delta_{\tilde{I}^a(-\vec{k})}, P_{\text{qu}} \right] \\ &= 2m P_{\text{qu}} - 4\pi m N \int d^2 k k^2 \alpha(k^2) \tilde{I}^a(\vec{k}) \delta_{\tilde{I}^a(\vec{k})} \quad , \end{aligned} \quad (15.22)$$

where the familar infinity is suppressed. Note that the latter would drop out in the wavy bracket of (15.19) anyway. Thus, $[\mathbf{T}, P_{\text{qu}}] 1 = 2m P_{\text{qu}}$. The doubly commutator gives

$$\frac{1}{2} \left[[\mathbf{T}, P_{\text{qu}}], P_{\text{qu}} \right] = -4\pi m N \int d^2 k k^2 \alpha^2(k^2) \tilde{I}^a(\vec{k}) \tilde{I}^a(-\vec{k}) \quad , \quad (15.23)$$

and (15.5) takes the form

$$0 = \int d^2 k \tilde{I}^a(\vec{k}) \tilde{I}^a(-\vec{k}) \left[\frac{1}{4\pi m N} + 2m \alpha(k^2) - 4\pi m N k^2 \alpha^2(k^2) \right] \quad . \quad (15.24)$$

As the square bracket must vanish, one obtains

$$\alpha(k^2) = \frac{m - \sqrt{m^2 + k^2}}{4\pi m N k^2} = -\frac{1}{4\pi m N} \frac{1}{m + \sqrt{m^2 + k^2}} \quad . \quad (15.25)$$

The second solution, having the opposite sign in front of the root, can be presumably ruled out towards normalizability. Inserting (15.25) into (15.21), the latter equation can be rewritten to be **{ 31 }**.

Having found the function $\alpha(k^2)$, not only P_{qu} is specified, but also the commutator $[\mathbf{T}, P_{\text{qu}}]$, (15.22). This in turn specifies the Hamiltonian \tilde{H} due (15.19). There we are :

$$\begin{aligned} \tilde{H} &= m \int d^2 k \tilde{I}^a(\vec{k}) \delta_{\tilde{I}^a(\vec{k})} + \mathbf{T}_2 - 4\pi m N \int d^2 k k^2 \alpha(k^2) \tilde{I}^a(\vec{k}) \delta_{\tilde{I}^a(\vec{k})} \\ &= \int d^2 k \sqrt{m^2 + k^2} \tilde{I}^a(\vec{k}) \delta_{\tilde{I}^a(\vec{k})} - \pi m N \int d^2 k k^2 \delta_{\tilde{I}^a(\vec{k})} \delta_{\tilde{I}^a(-\vec{k})} \quad . \end{aligned} \quad (15.26)$$

For the first excited state (in field-square approximation) we now might look for “the first Hermite polynomial”, i.e. the above \tilde{H} might be applied to a lonely I :

$$\tilde{H} \tilde{I}^a(\vec{k}) = \sqrt{m^2 + k^2} \tilde{I}^a(\vec{k}) \quad , \quad (15.27)$$

which exhibits the effect of \mathbf{V} to the formerly degenerated eigenvalue m . Note that index a and wave vector \vec{k} have converted to be quantum numbers of the wave functional. The corresponding eigenfunction of the original Hamiltonian H is $I e^{P_{\text{qu}}}$. Even

$\tilde{I}^a(\vec{k}_1)\tilde{I}^b(\vec{k}_2)e^{P_{\text{qu}}}(\vec{k}_2 \neq \pm \vec{k}_1)$ solves the stationary Schrödinger equation, with eigenvalue $\sqrt{m^2 + k_1^2} + \sqrt{m^2 + k_2^2}$, and so on. Without NT, one just arrives at a wave-functional version of a free massive ϕ^2 theory. But, unfortunately, this holds true only without NT. This nonlinear term, Ara Sedrakyan says, is the most important one.

Well. The root spectrum is very appealing. Also, it would be very welcome as the potential effect in lifting the degeneracy. But there is the NT. Could the effect of the NT be anyhow subdominant? (so noticed as an expectation at the end of [5]). The next step, including the NT, amounts to summing up the field-cubic terms in $P_1 + P_2 + P_3 + \dots$. KKN state the result in [5] as **{ 33 }**, **{ 34 }** (and we have verified). The expression shows that P reduces to P_1 at $k \rightarrow 0$, indeed. So, the sorting according to field powers is reasonable in the vacuum wave function and towards the Wilson loop analysis, see §§ 16, 17. But for obtaining the spectrum, we had no success (through (15.20) and some additional work) in recognising any subdominance of the NT. Further analysis is required. See also §18.3.

Perhaps, with respect to the excitation spectrum, we stand merely at the beginning. From naive physical imagination we would like to obtain *white* glue balls (as the lowest states), having some mass m_\bullet , and being in plane wave states with momentum \vec{k} and energy $\sqrt{m_\bullet^2 + k^2}$. For some first steps see § 6 of [1].

16 Wilson loop

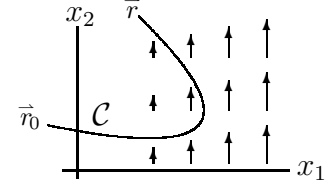
In non-abelian theory, where even the field tensor transforms, gauge invariant (hence physical) quantities are of particular high value. Moreover, the construction $W(\mathcal{C}) = \text{Tr} \left(\mathcal{P} e^{-\oint_{\mathcal{C}} d\vec{r} \vec{A}} \right)$ is even a continuum of gauge invariants, since the closed curve \mathcal{C} may be chosen at will. Everyone knows. Really? W is not a speciality of lattice theory. We follow the text book⁶ of Peskin and Schroeder [21], pedestrianize a bit and come to the restricted dependence of W on the WZW current in subsection 16.5. So, in this § 16, W remains a functional of fields (and of \mathcal{C} , too, of course). For the ground state expectation value $\langle W \rangle$ see § 17.

16.1 Potential on a curve

Mechanics. A force \vec{K} with $\nabla \times \vec{K} \neq 0$ has no potential. Since having a curl to the right, but none to the left, the equation “ $\nabla V = -\vec{K}$ ” is wrong — but “not very wrong”. Given a curve \mathcal{C} through the field of force, one may calculate the work done as $\int_{\mathcal{C}} ds \vec{r}_s \vec{K}$. On

⁶ Start chapter 15 of [21] from the beginning.

\mathcal{C} things become one-dimensional, and in 1D there is always a potential. Of course, it depends on the choice of the curve : $V(\vec{r}, \mathcal{C}, \vec{r}_0)$. Let s be the length on \mathcal{C} . Then $\vec{r}_{/s}$ is a variable unit vector, and $\vec{r}_{/s} \cdot \vec{K}$ the component of \vec{K} along the curve :



$$\partial_s V(\vec{r}(s), \mathcal{C}, \vec{r}_0) = \vec{r}_{/s} \cdot \nabla V = -\vec{r}_{/s} \cdot \vec{K}(\vec{r}(s)) \quad . \quad (16.1)$$

Hence, “ $\nabla V = -\vec{K}$ ” needs only the interpretation of providing all information, to be extracted by multiplication with an arbitrary (!) $\vec{r}_{/s}$, to the curve-dependent object V . Integration over s gives

$$V(\vec{r}(s), \mathcal{C}, \vec{r}_0) = V(\vec{r}_0, \mathbf{0}, \vec{r}_0) - \int_0^s ds' \vec{r}_{/s}(s') \cdot \vec{K}(\vec{r}(s')) \quad . \quad (16.2)$$

For the next, assume the force to be coupled (anyhow, by computer, say) to the potential by $\vec{K} = \vec{k} V$. Then “ $\nabla V = -\vec{k} V$ ” allows for the interpretation

$$\begin{aligned} \partial_s V(\vec{r}(s), \mathcal{C}, \vec{r}_0) &= - \left[\vec{r}_{/s} \cdot \vec{k}(\vec{r}(s)) \right] V(\vec{r}(s), \mathcal{C}, \vec{r}_0) \\ \text{with solution } V(\vec{r}(s), \mathcal{C}, \vec{r}_0) &= e^{-\int_0^s ds' \vec{r}_{/s}(s') \cdot \vec{k}(\vec{r}(s'))} V(\vec{r}_0, \mathbf{0}, \vec{r}_0) \quad , \end{aligned} \quad (16.3)$$

or, relaxing the notation :

$$V(\vec{r}, \mathcal{C}, \vec{r}_0) = \exp \left(- \int_{\mathcal{C}} d\vec{r} \cdot \vec{k} \right) V(\vec{r}_0, \mathbf{0}, \vec{r}_0) \quad . \quad (16.4)$$

By the next example (next subsection) we shall be led into field theory. For convenience, but other than in [21], we keep track with 2D Euclidean coordinates $(x_1, x_2) =: \vec{r}$, metrics $++$, two (times n) real fields A_1^a, A_2^a , antihermitean matrices $A_j := -iA_j^a T^a$, $(A_1, A_2) =: \vec{A}$ and with $(D_1, D_2) =: \vec{D}$. Herewith, the gauge transformation and the covariant derivative may be recapitulated as

$$\begin{aligned} \vec{A}^{[U]} &= U \vec{A} U^{-1} - (\nabla U) U^{-1} \quad , \quad U = e^{-i\Lambda^a T^a} \\ \vec{D} &= \nabla + \vec{A} \quad , \quad \vec{D}^{[U]} := \nabla + \vec{A}^{[U]} = U \vec{D} U^{-1} \quad . \end{aligned} \quad (16.5)$$

16.2 Wilson line

Let this subsection head give a name to the on- \mathcal{C} solution V of the “wrong” differential equation

$$“ \vec{D} V = 0 ” \quad , \quad \text{or} \quad “ \nabla V = -\vec{A} V ” \quad (16.6)$$

to a given initial condition $V(\vec{r}_0, \mathbf{0}, \vec{r}_0)$. Here, other than in “ $\nabla V = -\vec{k} V$ ”, \vec{A} as well as V are $N \times N$ matrices. Of course, if applied to an N -component spinor, (16.6) is nothing but the Dirac equation of motion of a massless quark field. The intermediate step

$$V(\vec{r}(s), \mathcal{C}, \vec{r}_0) = V(\vec{r}_0, \mathbf{0}, \vec{r}_0) - \int_0^s ds' \vec{r}_{/s}(s') \cdot \vec{A}(\vec{r}(s')) V(\vec{r}(s'), \mathcal{C}, \vec{r}_0) \quad (16.7)$$

shows that the V matrix keeps standing at the right end, even while iterating (16.7). Recognizing the time development in cases of a time dependent Hamiltonian, the solution to (16.7) can be dealt with as a path ordered (symbol \mathcal{P}) exponential :

$$V(\vec{r}, \mathcal{C}, \vec{r}_0) = \mathcal{P} e^{-\int_{\mathcal{C}} d\vec{r} \vec{A}} V(\vec{r}_0, \mathbf{0}, \vec{r}_0) \quad . \quad (16.8)$$

16.3 The Wilson loop

From (16.8) the curve- \mathcal{C} specific number W , called Wilson loop or Polyakov loop⁷, is obtained by the following three simple manipulations :

1. specify the initial condition as $V(\vec{r}_0, \mathcal{C}, \vec{r}_0) = 1$
2. close the curve \mathcal{C} : $\vec{r} = \vec{r}_0$
3. perform the trace of the so far $N \times N$ fold result :

$$W(\mathcal{C}) = \text{Tr} \left(\mathcal{P} e^{-\oint_{\mathcal{C}} d\vec{r} \vec{A}} \right) \quad . \quad (16.10)$$

This functional $W(\mathcal{C})$ is the one, which turns into itself by an arbitrary gauge transformation. Presumably, this invariance can be shown anyhow directly with the expression (16.10). But the pleasant and short way goes back to the differential equation that determines W uniquely. While retaining the unit initial condition, someone starts from (16.6) with a regauged field $\vec{A}^{[U]}$, calls his solution $V^{[U]}$ and omits quotation marks for brevity :

$$\begin{aligned} \nabla V^{[U]} &= -\vec{A}^{[U]} V^{[U]} \quad \text{to} \quad V_0^{[U]} = 1 \\ U \vec{D} U^{-1} V^{[U]} &= 0 \quad ; \quad U^{-1} V^{[U]} =: \mathcal{U} \\ \vec{D} \mathcal{U} &= 0 \quad \text{to} \quad \mathcal{U}_0 = U^{-1}(\vec{r}_0) \quad . \end{aligned} \quad (16.11)$$

Fortunately, in (16.8) the initial condition was kept arbitrary. So, the solution to (16.11) can be booked down immediately or, equivalently, $V^{[U]} = U(\vec{r}) \mathcal{U}(\vec{r})$:

$$V^{[U]}(\vec{r}, \mathcal{C}, \vec{r}_0) = U(\vec{r}) \mathcal{P} e^{-\int_{\mathcal{C}} d\vec{r} \vec{A}} U^{-1}(\vec{r}_0) \quad . \quad (16.12)$$

Now close the curve \mathcal{C} , take the trace and enjoy obtaining the number $W(\mathcal{C})$ of (16.10) again, *quod erat demonstrandum*. Yes, W is a functional of \mathcal{C} and of the fields A_1, A_2 . With respect to the A -fields, W separates between regaugings (no change) and the physical orbit. So far what “everyone knows”.

⁷ M. Thies [22] feels well with this term. Ara Sedrakyan knows of a corresponding manuscript of Polyakov. Obviously, the gauge invariant loop had been discovered by Polyakov and Wilson independently, such as it happened with the exact solution to the Kondo problem by Wiegmann and Andrei.

16.4 Reduction of $W(\mathcal{C})$ to physical degrees of freedom

For the 2+1 D Yang–Mills system we know more, namely the physical degrees ρ of freedom. Using $A = -(\partial M)M^{-1}$ and $M = U\rho$, we may exploit the gauge invariance just derived by simply turning to $U \equiv 1$ in the last of the following four lines. Remember that $z := x_1 - ix_2$, $\bar{z} := x_1 + ix_2$, $\frac{1}{2}(A_1 + iA_2) =: A$, $\frac{1}{2}(A_1 - iA_2) =: \bar{A} \rightsquigarrow A^\dagger = -\bar{A}$ and $\partial = \frac{1}{2}(\partial_1 + i\partial_2)$. Hence, the Wilson loop exponent may be rewritten as

$$\begin{aligned}
-\oint d\vec{r} \vec{A} &= \oint (-dz A - d\bar{z} \bar{A}) = \oint (-dz A + d\bar{z} A^\dagger) \\
&= \oint (dz (\partial M)M^{-1} - d\bar{z} M^{\dagger-1} \bar{\partial} M^\dagger) \\
&= \oint (dz (\partial U)U^\dagger + dz U(\partial\rho)\rho^{-1}U^\dagger - d\bar{z} U\rho^{-1}(\bar{\partial}\rho)U^\dagger - d\bar{z} U\bar{\partial}U^\dagger) \\
&= \oint (dz (\partial\rho)\rho^{-1} - d\bar{z} \rho^{-1}\bar{\partial}\rho) \quad . \tag{16.13}
\end{aligned}$$

Of course, the last step, namely setting $U \equiv 1$, is only allowed after the exponent is already placed into the traced and ordered exponential function :

$$W(\mathcal{C}) = \text{Tr} \left(\mathcal{P} e^{\oint (dz (\partial\rho)\rho^{-1} - d\bar{z} \rho^{-1}\bar{\partial}\rho)} \right) \quad . \tag{16.14}$$

16.5 $W(\mathcal{C})$ is a functional of only J

The exponent $-\oint_{\mathcal{C}} d\vec{r} \vec{A}$ of (16.10) was an antihermitean matrix, as is the exponent of (16.14), of course. But this property has gone in KKN's expression [2.32] (\equiv (16.21) below) for $W(\mathcal{C})$. How this? To tackle this problem, we might go back again to the differential equation, but this time to that of the physical version (16.14).

It may well be guessed how (16.14) might derive from differential equations, namely from the following “wrong” pair :

$$(\partial - (\partial\rho)\rho^{-1}) V_{\text{ph}} = 0 \quad , \quad (\bar{\partial} + \rho^{-1}(\bar{\partial}\rho)) V_{\text{ph}} = 0 \quad \text{to} \quad V_{\text{ph}0} = 1 \quad . \tag{16.15}$$

But now go slowly, dear pedestrian, to enjoy the details. At first, the line (16.15) is a bit rewritten for harmony :

$$\partial \rho^{-1} V_{\text{ph}} = 0 \quad , \quad \bar{\partial} \rho V_{\text{ph}} = 0 \quad \text{to} \quad V_{\text{ph}0} = 1 \quad . \tag{16.16}$$

From this we are led to the idea of introducing ρV_{ph} as a new Wilson–line matrix. Also, this brings the matrix $H = \rho^2$ into play :

$$\rho V_{\text{ph}} =: \Upsilon \quad : \quad \partial H^{-1} \Upsilon = 0 \quad , \quad \bar{\partial} \Upsilon = 0 \quad \text{to} \quad \Upsilon_0 = \rho(\vec{r}_0) \quad . \tag{16.17}$$

Writing this as

$$\partial \Upsilon = (\partial H) H^{-1} \Upsilon \quad , \quad \bar{\partial} \Upsilon = 0 \quad \text{to} \quad \Upsilon_0 = \rho(\vec{r}_0) \quad , \tag{16.18}$$

(16.8) can be applied once more to get the solution

$$\Upsilon(\vec{r}, \mathcal{C}, \vec{r}_0) = \mathcal{P} e^{\int_{\mathcal{C}} dz (\partial H) H^{-1}} \rho(\vec{r}_0) \quad , \quad (16.19)$$

having no $d\bar{z}$ term any more. To form $W(\mathcal{C})$ we need $V_{\text{ph}} = \rho^{-1}(\vec{r}) \Upsilon(\vec{r})$,

$$V_{\text{ph}}(\vec{r}, \mathcal{C}, \vec{r}_0) = \rho^{-1}(\vec{r}) \mathcal{P} e^{\int_{\mathcal{C}} dz (\partial H) H^{-1}} \rho(\vec{r}_0) \quad . \quad (16.20)$$

Close the curve \mathcal{C} , take the trace — and, surprisingly, KKN are right with their mysterious statement [2.32] :

$$W(\mathcal{C}) = \text{Tr} \left(\mathcal{P} e^{\oint dz (\partial H) H^{-1}} \right) = \text{Tr} \left(\mathcal{P} e^{\frac{\pi}{N} \oint dz J} \right) \quad . \quad (16.21)$$

The Wilson loop, reduced to physical variables, is a functional of only (apart from \mathcal{C}) the WZW current J or, equivalently, of $J^a = 2 \text{Tr} (T^a J) = \frac{N}{\pi} 2 \text{Tr} (T^a (\partial H) H^{-1})$.

The wisdom (16.21) has now to be combined with an other one, namely that, according to [21], *in fact, all gauge-invariant functions of A^μ can be thought of as combinations of Wilson loops for various choices of the path \mathcal{P}* . For a few more explanations see [21]. The combination of this argument with (16.21) means, that any physics (including normalizable ψ 's) might have functional dependence of only J .

17 Confinement

Sometimes, the “well known” things make the trouble, while the advanced matter can be done merely straightforwardly. Here it is the first subsection § 17.1, which made the problems — answered more or less good, hopefully. Then, through the other subsections, $\langle W \rangle$ is calculated explicitly for a large loop \mathcal{C} . — Why $\langle W \rangle$?

17.1 Two-quark state

Let us first state what we need. If the pure-gluonic full-interacting-groundstate expectation value $\langle 0 | W(\mathcal{C}) | 0 \rangle = \langle W \rangle_{\psi_0}$, or $\langle W \rangle$ for brevity, can be analysed in the limit of large area \mathbf{A} surrounded by the closed (non-intersecting) curve \mathcal{C} , and if this leads to the “area law”

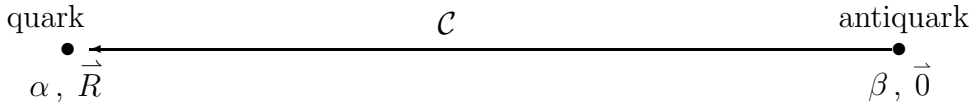
$$\langle 0 | W(\mathcal{C}) | 0 \rangle =: \langle W \rangle \xrightarrow{\text{both } \mathbf{A}\text{-dimensions} \rightarrow \infty} e^{-\sigma \mathbf{A}} \quad , \quad (17.1)$$

then there is “confinement”, σ is the “string tension” and the energy E_0 in the gluon system, which surrounds and connects a fixed quark–antiquark pair at large separation R , increases as $E_0 = \sigma R$. Let the area \mathbf{A} lie in a plane. But it is irrelevant, whether this plane is fully spatial or extends in time direction, because the latter is Euclidean anyway. Show all this.

To keep a quark fixed at (or near) the position \vec{R} (in 3-dimensional space, say) we need hypothetical “other” forces. The quark being bounded at \vec{R} , its field operator $\bar{q}_\alpha(\tau, \vec{r})$ contains some real function $\psi_{\vec{R}}(\vec{r})$, which is strongly localized around \vec{R} and takes the place of the plane wave, times b_α^\dagger , the quark creator, times γ^0 . Conveniently, we shall rather state its center \vec{R} in place of the variable \vec{r} , in the sequel. The time is rotated into the Euclidean by $t \rightarrow -i\tau$. Let also an antiquark be fixed at the origin with field operator $q_\beta(\tau, \vec{r})$, containing some function $\psi_{\vec{0}}$ times d_β^\dagger , the antiquark creator. We follow Bander [23] (a delicate article of 1981, blind copies from are found in two text books) to write down a gauge invariant “meson state” as

$$| \text{meson at } \tau \rangle = \bar{q}_\alpha(\tau, \vec{R}) V_{\alpha\beta}(\tau, \vec{R}, \mathcal{C}, \tau, \vec{0}) q_\beta(\tau, \vec{0}) |0\rangle =: \Gamma(\tau) |0\rangle \quad (17.2)$$

Here, $|0\rangle$ is a bare quark vacuum, and V is the non-abelian object (16.8) with matrix indices made explicit (and suitably generalized to include time). For the present, the gauge fields in V are considered classical.



There are two circumstances in favour of the expression (17.2). First of all, it is gauge invariant, indeed, since according to (16.12) the U matrices recombine with those splitting off from the quark fields at the right space-time points. Secondly, imagine a quark-antiquark pair produced at the origin and the quark then be moved to \vec{R} along the path \mathcal{C} . Thereby, as we know from nonrelativistic quantum mechanics in given classical fields (see Aharonov Bohm effect), it develops just the phase shown in (17.2). Admittedly, since depending on \mathcal{C} , the term “meson state” needs its parantheses. A sum (sum???) over paths is seen in [23] together with some unspecified weight function of \mathcal{C} . But please, Dr. Bander, one cannot state this sum on one page and simply omit it on the next. Presumably, this detail (this arbitrariness) becomes irrelevant in the limit of large separation. We shall keep track with a straight line connecting the quark pair.

So far, there were no conclusions, hence all the above is $=:$ right. Furthermore, we may consider what we want. Consider the overlap

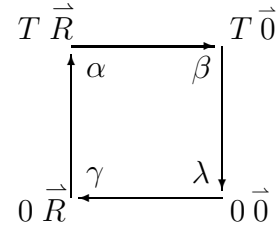
$$\begin{aligned} \Omega_{\text{classical}} &:= \langle \text{meson at } T \mid \text{meson at } 0 \rangle = \langle 0 \mid \Gamma^\dagger(T) \Gamma(0) \mid 0 \rangle \\ &= \langle 0 \mid q_\beta^\dagger(T, \vec{0}) V_{\alpha\beta}^*(T, \vec{R}, \mathcal{C}, T, \vec{0}) \bar{q}_\alpha^\dagger(T, \vec{R}) \bar{q}_\gamma(0, \vec{R}) V_{\gamma\lambda}(0, \vec{R}, \mathcal{C}, 0, \vec{0}) q_\lambda(0, \vec{0}) \mid 0 \rangle \\ &= \langle 0 \mid \bar{q}_\beta(T, \vec{0}) V_{\beta\alpha}(T, \vec{0}, \mathcal{C}, T, \vec{R}) q_\alpha(T, \vec{R}) \bar{q}_\gamma(0, \vec{R}) V_{\gamma\lambda}(0, \vec{R}, \mathcal{C}, 0, \vec{0}) q_\lambda(0, \vec{0}) \mid 0 \rangle \\ &= \langle 0 \mid \bar{q}_\beta(T, \vec{0}) q_\lambda(0, \vec{0}) \mid 0 \rangle \langle 0 \mid q_\alpha(T, \vec{R}) \bar{q}_\gamma(0, \vec{R}) \mid 0 \rangle V_{\beta\alpha}(T, \vec{0}, \mathcal{C}, T, \vec{R}) V_{\gamma\lambda}(0, \vec{R}, \mathcal{C}, 0, \vec{0}) \\ &= \langle 0 \mid \bar{q}_\sigma(0, \vec{0}) q_\lambda(0, \vec{0}) \mid 0 \rangle e^{-mT} V_{\sigma\beta}(0, \vec{0}, \mathcal{C}, T, \vec{0}) \\ &\quad \langle 0 \mid q_\alpha(T, \vec{R}) \bar{q}_\rho(T, \vec{R}) \mid 0 \rangle e^{-mT} V_{\rho\gamma}(T, \vec{R}, \mathcal{C}, 0, \vec{R}) V_{\beta\alpha}(T, \vec{0}, \mathcal{C}, T, \vec{R}) V_{\gamma\lambda}(0, \vec{R}, \mathcal{C}, 0, \vec{0}) \\ &\sim e^{-2mT} V_{\lambda\beta}(0, \vec{0}, \mathcal{C}, T, \vec{0}) V_{\alpha\gamma}(T, \vec{R}, \mathcal{C}, 0, \vec{R}) V_{\beta\alpha}(T, \vec{0}, \mathcal{C}, T, \vec{R}) V_{\gamma\lambda}(0, \vec{R}, \mathcal{C}, 0, \vec{0}) \end{aligned} \quad (17.3)$$

Third line. The c.c. of V is the same with reversed matrix indices and reversed curve (exercise). Also, $(q_\alpha^\dagger \gamma^0)^\dagger = \gamma^0 q_\alpha$ had been used. Do not smile. Rather remember that in q_a^\dagger the term $\sim \psi_{\vec{R}}^\dagger b_\alpha^\dagger$ was relevant, and this has now turned into $\sim \psi_{\vec{R}}^\dagger b_\alpha$. Hence, $q_\alpha(T\vec{R})$ in the third line annihilates a quark at \vec{R} . Moving γ^0 to the left, \bar{q}_β is formed. It annihilates an antiquark at origin.

Fourth line. Here, Wicks theorem has only one term. The second term vanishes, since there is no overlap between the localizing functions ψ at different positions. Both the two bare quark propagators contain an annihilator to the left and a creator to the right.

Fifth line. $q_\beta(0\vec{0}) \sim \psi_{\vec{0}}^\dagger d_\beta^\dagger$ had created the antiquark. In the presence of classical fields it develops as $V_{\beta\sigma}(T\vec{0}, \mathcal{C}, 0\vec{0}) q_\sigma(0\vec{0})$, or daggered as $\bar{q}_\beta(T\vec{0}) = \bar{q}_\sigma(0\vec{0}) V_{\sigma\beta}(0\vec{0}, \mathcal{C}, T\vec{0}) e^{-mT}$, where the exponential stems from $e^{iH_0 t} d_\beta e^{-iH_0 t} = e^{-imt} d_\beta \rightarrow e^{-mT} d_\beta$. It is, by the way, irrelevant towards the area law.

Last line. Here, only the Kroneckers $\delta_{\sigma\lambda}$ and $\delta_{\alpha\beta}$ are retained from the propagators. The four Wilson lines combine to the square shown, which is a special Wilson loop in the x - τ plane, say.



So far the gauge fields have remained classical. But now let them become operators and (17.3) be put in the full gluonic ground state expectation value. The two ground states, $|0\rangle_{\text{gluon}}$ and the bare quark vacuum $|0\rangle$, may be combined to a product state $|0_\bullet\rangle$. The reason for the sandwiching is seen as follows :

$$\begin{aligned}
 \Omega &:= \langle 0|_{\text{gluon}} \Omega_{\text{classical}} |0\rangle_{\text{gluon}} = \langle 0_\bullet | \Gamma^\dagger(T) \Gamma(0) |0_\bullet \rangle \\
 &= \sum_n \langle 0_\bullet | \Gamma^\dagger(T) |n_\bullet \rangle \langle n_\bullet | \Gamma(0) |0_\bullet \rangle = \sum_n e^{-E_\bullet n T} |\langle n_\bullet | \Gamma(0) |0_\bullet \rangle|^2 \\
 &\sim e^{-E_\bullet 0 T} = e^{-2mT - E_0 T} \quad (T \rightarrow \infty) ,
 \end{aligned} \tag{17.4}$$

where E_0 is the gluonic part of the meson (i.e. the energy in the *string* of field lines, if this picture makes sense at all). Note that, as Γ creates and Γ^\dagger annihilates the whole meson, we have $e^{iHt} \Gamma^\dagger(0) e^{-iHt} = e^{-iEt} \Gamma^\dagger(0) \rightarrow e^{-E\tau} \Gamma^\dagger(0)$.

If the gluon energy increases as $E_0 = \sigma R$ with large spatial separation R , then the exponent $-E_0 T$ turns into $-\sigma R T = -\sigma \mathfrak{A}$, as announced. Forming the gluon sandwich with the last line of (17.3), Ω may be also written as

$$\Omega \sim e^{-2mT} \langle W(\mathcal{C}) \rangle . \tag{17.5}$$

By comparison of (17.5) with (17.4), the statement (17.1) is derived. Really ?

17.2 A special 2D Euclidean J theory with action \mathfrak{S}

We return to the 2+1 D Yang–Mills system in Weyl gauge. As in § 16, the closed loop \mathcal{C} lies in the 2D xy -plane. We trust in (17.1) and the statements around it. In wave quantum

mechanics, the ground state expectation value (17.1) must be written as a functional integral over physical variables, i.e. with the measure $d\mu(\mathcal{C})$. Oh, two \mathcal{C} 's with different meaning. We change the notation of the measure from $d\mu(\mathcal{C})$ to $d\mu_{\text{phys}}$:

$$\langle W \rangle = \langle 0 | W(\mathcal{C}) | 0 \rangle = \int d\mu_{\text{phys}} \psi_0 W(\mathcal{C}) \psi_0 = \int d\mu_{\text{phys}} e^{2P} W(\mathcal{C}) \quad . \quad (17.6)$$

Here $\psi_0[J] = e^{P[J]} 1$ is the exact ground state functional, as introduced in (15.2) together with the functional P (exact as well). ψ_0 is chosen real.

In the limit of large area in \mathcal{C} (large in both directions), we may argue that the functional P is probed at short wavevectors only. This is just the limiting case that could be worked out in § 15.1 with the result (15.11) :

$$2P \rightarrow 2P_1 = -\frac{\mathbf{V}}{m} = -\frac{\pi}{m^2 N} \int (\bar{\partial} J^a) \bar{\partial} J^a = -\frac{2\pi}{m^2 N} \int \text{Tr}((\bar{\partial} J) \bar{\partial} J) =: -\$ \quad . \quad (17.7)$$

Remember that $J = J^a T^a$. Reading $\$$ as one more auxiliary 2D action, but this time with the currents J^a as fields and containing no interaction, the Wilson loop expectation value $\langle W \rangle$ becomes a functional integral of a free theory, namely :

$$\langle W(\mathcal{C}) \rangle = \int d\mu_{\text{phys}} e^{-\frac{2\pi}{m^2 N} \int \text{Tr}((\bar{\partial} J) \bar{\partial} J)} \text{Tr} \left(\mathcal{P} e^{\frac{\pi}{N} \$ dz J} \right) \quad . \quad (17.8)$$

Its evaluation might be anyhow possible. Note that the WZW action S (KKN should distinguish $\$$ from S) is quite another object. It contains no derivatives of J and is, via e^{2NS} , part of the measure $d\mu_{\text{phys}}$ in the above.

17.3 The J propagator

We shall need the propagator of the $\$$ theory for matrix currents $J_{\alpha\beta}(\vec{r})$, path ordered along the curve \mathcal{C} :

$$\begin{aligned} \langle \mathcal{P} J_{\alpha\beta}(\vec{r}) J_{\lambda\tau}(\vec{r}') \rangle &= T_{\alpha\beta}^a T_{\lambda\tau}^b \left[\theta_{\vec{r} > \vec{r}'} \underbrace{\langle J^a(\vec{r}) J^b(\vec{r}') \rangle}_{\substack{\text{on } \mathcal{C} \\ \text{on } \mathcal{C}}} + \theta_{\vec{r} < \vec{r}'} \langle J^b(\vec{r}') J^a(\vec{r}) \rangle \right] \\ &= T_{\alpha\beta}^a T_{\lambda\tau}^a \Delta(\vec{r} - \vec{r}') \stackrel{!}{=} \delta^{ab} \Delta(\vec{r} - \vec{r}') \\ &= \frac{1}{2} \left(\delta_{\alpha\tau} \delta_{\beta\lambda} - \frac{1}{N} \delta_{\alpha\beta} \delta_{\lambda\tau} \right) \Delta(\vec{r} - \vec{r}') \quad , \end{aligned} \quad (17.9)$$

because $\Delta(\vec{r}) = \Delta(-\vec{r})$ will turn out shortly. The average in the first line is defined by $\langle \dots \rangle := \int d\mu_{\text{phys}} e^{-\$} \dots$. For the function Δ we first Fourier transform the action by inserting $J^a = \int \frac{d^2 k}{(2\pi)^2} e^{i\vec{k} \cdot \vec{r}} \tilde{J}^a(\vec{k})$ and introduce the propagator $\tilde{\Delta}^{ab}(\vec{k})$ by the last line :

$$\begin{aligned} \$ &= \frac{\pi}{m^2 N} \int (\bar{\partial} J^a) \bar{\partial} J^a = \frac{\pi}{4m^2 N} \int \frac{d^2 k}{(2\pi)^2} \tilde{J}^a(\vec{k}) (k_1 - ik_2)^2 \tilde{J}^a(-\vec{k}) \\ &\stackrel{!}{=} -\frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \tilde{J}^a(\vec{k}) \left(\frac{1}{\tilde{\Delta}(\vec{k})} \right)^{ab} \tilde{J}^b(-\vec{k}) \quad . \end{aligned} \quad (17.10)$$

Clearly, the Fourier transformed propagator is given by

$$\tilde{\Delta}^{ab}(\vec{k}) = \delta^{ab} \tilde{\Delta}(\vec{k}) \quad \text{with} \quad \tilde{\Delta}(\vec{k}) = \frac{2m^2 N}{\pi} \frac{-1}{(k_1 - ik_2)^2} . \quad (17.11)$$

The transformation back to real space needs no additional work, since it runs along the five lines of (13.21). Just k_1, k_2 are to be interchanged with x, y , and an additional factor of $1/(2\pi)^2$ must be included here. Hence, the result is

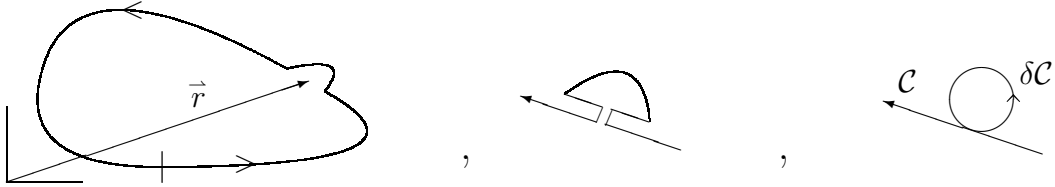
$$\Delta^{ab}(\vec{r}) = \delta^{ab} \Delta(\vec{r}) \quad \text{with} \quad \Delta(\vec{r}) = \frac{m^2 N}{2\pi^2} \frac{x + iy}{x - iy} = \Delta(-\vec{r}) . \quad (17.12)$$

It specifies (17.9).

17.4 Varying the area : $\partial_{\mathbf{A}} \langle W \rangle$

For obtaining the desired differential equation for $\langle W \rangle$ a paper of Kasakov and Kostov [24] was useful. Possibly, Gross [25] knows of another shorter way to derive the area law. But [25] is so far not understood with respect to both, prerequisites and stability. We favour explicit calculation.

Increasing the area \mathbf{A} means elongating its border line \mathcal{C} , e.g. by an infinitesimal bump at position \vec{r} on \mathcal{C} . Hence, the derivative $\partial_{\mathbf{A}(\vec{r})}$ carries \vec{r} as an index. But the result will, of course, not depend on this position. Moreover, the bump can be given any convenient form. The circle version to the right in the figure turned out most convenient.



Let us write $\mathcal{C} + \delta\mathcal{C}$ for the enlarged curve. Of course, in the differentiation to be performed,

$$\partial_{\mathbf{A}} \langle W \rangle = \frac{1}{\delta\mathbf{A}} \langle W(\mathcal{C} + \delta\mathcal{C}) - W(\mathcal{C}) \rangle \quad , \quad W(\mathcal{C}) = \text{Tr} \left(\mathcal{P} e^{\frac{\pi}{N} \oint_{\mathcal{C}} dz J} \right) \quad , \quad (17.13)$$

the path ordering runs through to the circle, if any. Due to the ordering \mathcal{P} in front of the exponential, we may relax inside and write $\oint_{\mathcal{C} + \delta\mathcal{C}} = \oint_{\mathcal{C}} + \oint_{\delta\mathcal{C}}$. Keeping $dz \frac{\pi}{N} J$ in mind, we may even write

$$e^{\oint_{\mathcal{C} + \delta\mathcal{C}}} - e^{\oint_{\mathcal{C}}} = \left[e^{\oint_{\delta\mathcal{C}}} - 1 \right] e^{\oint_{\mathcal{C}}} = \left[\oint_{\delta\mathcal{C}} + \frac{1}{2} \oint_{\delta\mathcal{C}} \oint_{\delta\mathcal{C}} \right] e^{\oint_{\mathcal{C}}} . \quad (17.14)$$

Note that $\mathbf{A} \sim \delta r^2$, with δr the radius of the circle. But it is $\oint_{\delta\mathcal{C}} \sim \delta r$, only. So, the second term in (17.14) must be retained. Correspondingly, there are two contributions :

$\partial_{\mathbf{A}} \langle W \rangle =: \partial_{\mathbf{A}} \langle W \rangle \Big|_1 + \partial_{\mathbf{A}} \langle W \rangle \Big|_2$. More explicitly they read

$$\partial_{\mathbf{A}} \langle W \rangle \Big|_1 = \frac{1}{\delta \mathbf{A}} \frac{\pi}{N} \oint_{\delta \mathcal{C}} dz \operatorname{Tr} \left(\left\langle \mathcal{P} J(\vec{r}) e^{\frac{\pi}{N} \oint_{\mathcal{C}} dz' J(\vec{r}')} \right\rangle \right) \quad (17.15)$$

$$\partial_{\mathbf{A}} \langle W \rangle \Big|_2 = \frac{1}{\delta \mathbf{A}} \frac{\pi^2}{2N^2} \oint_{\delta \mathcal{C}} dz \oint_{\delta \mathcal{C}} dz' \operatorname{Tr} \left(\left\langle \mathcal{P} J(\vec{r}) J(\vec{r}') e^{\frac{\pi}{N} \oint_{\mathcal{C}} dz'' J(\vec{r}'')} \right\rangle \right) . \quad (17.16)$$

As an outlook, the second contribution (17.16) will in fact be argued to vanish for the regularised (!) theory at hand (but not otherwise). We shall understand this in the course of evaluating the first contribution.

Imagine the exponential of (17.15) be expanded. The functional average $\langle \quad \rangle$ will “pair” each product of J ’s, i.e. decompose it into a sum of propagators $\langle J J \rangle$ (Wick’s theorem in functional integral language). With this fact in mind, we may even leave the exponential intact. To pair the extra $J(\vec{r})$ in all possible ways with one in the exponential we put the operator

$$\int d^2 r_0 J_{\lambda\tau}(\vec{r}_0) \delta_{J_{\lambda\tau}(\vec{r}_0)} \quad (17.17)$$

in front of it. Note that it selects each inner J once. Hence

$$\begin{aligned} \partial_{\mathbf{A}} \langle W \rangle \Big|_1 &= \frac{1}{\delta \mathbf{A}} \frac{\pi}{N} \oint_{\delta \mathcal{C}} dz \int d^2 r_0 \left\langle \mathcal{P} J_{\alpha\beta}(\vec{r}) J_{\lambda\tau}(\vec{r}_0) \right\rangle \\ &\quad \cdot \left\langle \delta_{J_{\lambda\tau}(\vec{r}_0)} \left(\mathcal{P} e^{\frac{\pi}{N} \oint_{\mathcal{C}} dz' J(\vec{r}')} \right)_{\beta\alpha} \right\rangle . \end{aligned} \quad (17.18)$$

The first average is just the matrix propagator as detailed in (17.9). Performing the functional differentiation in the second average, is left as an exercise to the reader (expand the exponential up to the third order term, at least. Better way?). But let us consider the first step into it, namely the first order term of the expansion :

$$\delta_{J_{\lambda\tau}(\vec{r}_0)} \frac{\pi}{N} \int_0^{\text{end}} dz' J_{\beta\alpha}(\vec{r}') = \frac{\pi}{N} \delta_{\lambda\beta} \delta_{\tau\alpha} \int_0^{\text{end}} dz' \delta(\vec{r}_0 - \vec{r}') . \quad (17.19)$$

This integral, removing “half of” the 2-dimensional delta function, occurs once in each of the other terms, too. The end point along the closed curve \mathcal{C} is, of course, identical with the point $\vec{0}$. The exercise results in

$$\begin{aligned} \delta_{J_{\lambda\tau}(\vec{r}_0)} \left(\mathcal{P} e^{\frac{\pi}{N} \oint_{\mathcal{C}} dz' J(\vec{r}')} \right)_{\beta\alpha} &= \frac{\pi}{N} \int_0^{\text{end}} dz'' \delta(\vec{r}_0 - \vec{r}'') \\ &\quad \left(\mathcal{P} e^{\frac{\pi}{N} \int_{\vec{r}_0}^{\text{end}} dz' J(\vec{r}')} \right)_{\beta\lambda} \left(\mathcal{P} e^{\frac{\pi}{N} \int_0^{\vec{r}_0} dz' J(\vec{r}')} \right)_{\tau\alpha} \end{aligned} \quad (17.20)$$

In the intermediate result, to be booked down next, we get rid of the above delta function through integrating over $d^2 r_0$, see (17.18). Then, after inserting (17.9) and using the Kroneckers there, we arrive at

$$\begin{aligned} \partial_{\mathbf{A}} \langle W \rangle \Big|_1 &= \frac{1}{\delta \mathbf{A}} \frac{1}{2} \left(\frac{\pi}{N} \right)^2 \oint_{\delta \mathcal{C}} dz \int_0^{\text{end}} dz'' \Delta(\vec{r} - \vec{r}'') . \\ &\quad \cdot \left\langle \left(\mathcal{P} e^{\text{from } \vec{r}''} \right)_{\beta\beta} \left(\mathcal{P} e^{\text{to } \vec{r}''} \right)_{\alpha\alpha} - \frac{1}{N} \left(\mathcal{P} e^{\text{from } \vec{r}''} \right)_{\alpha\lambda} \left(\mathcal{P} e^{\text{to } \vec{r}''} \right)_{\lambda\alpha} \right\rangle . \end{aligned} \quad (17.21)$$

The two matrices at the right end may be combined to $\text{Tr}(\mathcal{P}e^{\cdots}) = W(\mathcal{C})$, immediately.

The doubly integral in (17.21),

$$\mathcal{N} := \oint_{\delta\mathcal{C}} dz \int_0^{\text{end}} dz'' \Delta(\vec{r} - \vec{r}'') \Big|_{\text{both, } \vec{r} \text{ and } \vec{r}'', \text{ on } \mathcal{C}} \quad , \quad (17.22)$$

will turn out $\sim \delta r^2$ in the next subsection. This means, that only some infinitesimal short range on \mathcal{C} makes \vec{r}'' different from \vec{r} . Hence, in the second line of (17.21), we are allowed to replace \vec{r}'' by \vec{r} , the position where we differentiate with respect to \mathbf{A} . Now remember that the start position $\vec{0}$ on the closed curve could have been chosen at will. Taking it near to \vec{r} we get $(\quad)_{\alpha\alpha} \rightarrow 1_{\alpha\alpha} = N$ and $(\quad)_{\beta\beta} \rightarrow W(\mathcal{C})$, in the second line of (17.21)⁸. So, up to the evaluation of \mathcal{N} , we have that

$$\partial_{\mathbf{A}} \langle W \rangle \Big|_1 = \frac{1}{\delta\mathbf{A}} \frac{1}{2} \left(\frac{\pi}{N} \right)^2 \mathcal{N} \frac{n}{N} \langle W \rangle \quad . \quad (17.23)$$

In fact, evaluating the prefactor \mathcal{N} means calculating the string tension.

17.5 The integral for the string tension

To start with, insert the Greens function (17.12) into (17.22),

$$\mathcal{N} =: \frac{m^2 N}{2\pi^2} \mathcal{N}_0 \quad , \quad \mathcal{N}_0 = \oint_{\delta\mathcal{C}} dz \int dz' \frac{(x - x') + i(y - y')}{(x - x') - i(y - y')} \quad , \quad (17.24)$$

and let the relevant piece of the curve \mathcal{C} be a vertical, parallel to the y -axis with distance δr . With the center of the circle $\delta\mathcal{C}$ at origin, the four cartesian variables may be parametrized as

$$\begin{aligned} x &= \delta r \cos(\varphi) & y &= \delta r \sin(\varphi) & dz &= dx - idy = -i\delta r e^{-i\varphi} d\varphi \\ x' &= -\delta r & y' &= \delta r t & dz' &= dx' - idy' = -i\delta r dt \end{aligned} \quad (17.25)$$

to give

$$\mathcal{N}_0 = (-i\delta r)^2 \int_{-\pi}^{\pi} d\varphi e^{-i\varphi} \int_{-\infty}^{\infty} dt \frac{e^{i\varphi} + 1 - it}{e^{-i\varphi} + 1 + it} \quad (17.26)$$

There are several possibilities of further evaluation. We set $1 + it =: \sqrt{1 + t^2} e^{i\alpha}$ and shift $\varphi \rightarrow \varphi - \alpha$:

$$\begin{aligned} \mathcal{N}_0 &= -(\delta r)^2 \int dt e^{-i\alpha} \int_{(2\pi)} d\varphi \frac{1 + \sqrt{1 + t^2} e^{-i\varphi}}{e^{-i\varphi} + \sqrt{1 + t^2}} \\ &= -(\delta r)^2 \int dt e^{-i\alpha} \int_{(2\pi)} d\varphi \left(\sqrt{1 + t^2} - \frac{t^2}{e^{-i\varphi} + \sqrt{1 + t^2}} \right) \\ &= -\frac{\delta\mathbf{A}}{\pi} \int dt e^{-i\alpha} \left(2\pi\sqrt{1 + t^2} - t^2 \frac{2\pi}{\sqrt{1 + t^2}} \right) \quad (*) \\ \mathcal{N}_0 &= -2\delta\mathbf{A} \int dt \frac{1 - it}{1 + t^2} = -2\pi\delta\mathbf{A} \quad , \end{aligned} \quad (17.27)$$

⁸ Admittedly, we do not like such dangerous arguments. Is there a better way?

where the line $(*)$ was reached by contour integration with $e^{i\varphi} \equiv z$ and a pole at $-1/\sqrt{1+t^2}$ inside the unit circle.

\mathcal{N} is proportional to the area $\delta\mathbf{A}$, indeed. But it specifies the first contribution $\partial_{\mathbf{A}} \langle W \rangle \Big|_1$ only. Let us now think about the fate of \mathcal{N} under regularization of the theory. Short distances might be smoothed. So we add a length $1/\Lambda$ to both, the numerator and the denominator of the propagator in (17.24). The calculation runs through the above steps and results in

$$\mathcal{N}_0^{\text{regularised}} = -2 \delta\mathbf{A} \int dt \frac{1 + \lambda - it}{(1 + \lambda)^2 + t^2} = -2 \pi \delta\mathbf{A} \quad \text{with} \quad \lambda := \frac{1}{\Lambda \delta r} \quad . \quad (17.28)$$

Thus, regularization (of the above kind, at least) has no effect on \mathcal{N} , although the factor λ tends to infinity under $\delta r \rightarrow 0$. Note that by omitting the propagator in (17.24) entirely ($\delta r \rightarrow 0$ before integrating), we would have $\oint_{\delta\mathcal{C}} dz \int dz' = "0 \cdot \infty"$.

For the second term, which is $\partial_{\mathbf{A}} \langle W \rangle \Big|_2$ and given by (17.16), a similar analysis leads one to consider

$$\oint_{\delta\mathcal{C}} dz \oint_{\delta\mathcal{C}} dz' \frac{(x - x') + i(y - y')}{(x - x') - i(y - y')} = \dots = 4\pi \delta\mathbf{A} \quad (17.29)$$

in place of \mathcal{N}_0 (more precisely, in place of $2n\mathcal{N}_0/N^2$). But the fate of this expression under regularization is quite different. It simply vanishes for $\lambda \rightarrow \infty$. For this, after all, the rough argument $\oint_{\delta\mathcal{C}} dz \oint_{\delta\mathcal{C}} dz' = "0 \cdot 0"$ might be sufficient. We might state, finally, that [24] (e.g. in treating intersecting curves \mathcal{C}) goes far ahead over what we have used here.

17.6 Area law

There we are. Combining (17.23), (17.24) and (17.27) the differential equation for the gluon-averaged Wilson loop turns about as

$$\partial_{\mathbf{A}} \langle W(\mathcal{C}) \rangle = -\frac{\pi n}{2N^2} m^2 \langle W(\mathcal{C}) \rangle \quad . \quad (17.30)$$

The initial condition is $\langle W(\mathcal{C} = 0) \rangle = 1$. Will we be able to solve this?

$$\langle W(\mathcal{C}) \rangle = e^{-\frac{\pi n}{2N^2} m^2 \mathbf{A}} \quad . \quad (17.31)$$

The string tension σ of 2+1 D QCD is thus given by

$$\sigma = \frac{\pi n}{2N^2} m^2 = e^4 \frac{N^2 - 1}{8\pi} \quad . \quad (17.32)$$

This is, we think, an exact result, because the approximation (17.7) used for the vacuum wave function ψ_0 becomes accurate for long wavelengths. Afterwards there were no further approximations towards a large loop. (17.32) is **{ 26 }**. In [5], the comparison with lattice data exhibits excellent agreement.

18 Magnetic mass

There is 4D reality, where all knowledge on the 3D YM system gets application. Moreover, for the high-temperature gluon gas it solves the outstanding Linde problem. So, it is quite natural, to adress a final section to this detail of thermal field theory. However, if continuing with the foregoing pedestrianity, a text book would arise. We therefore decide for the other extreme, refer to our own material in § 18.1, add a few general remarks in § 18.2 and end up with a hairraising conjecture in § 18.3.

18.1 The Linde sea

[18], *Magnetic screening in the hot gluon system*, Introduction :

Twenty years ago it was observed by Linde [2] that the perturbative treatment of the high-temperature Yang-Mills system runs into a serious problem. If a magnetic mass m , the system might be able to generate thermally, falls short of g^2T in magnitude, the series would diverge, and a phase of deconfined gluons could not exist. But even if $m \sim g^2T$, the perturbation series becomes an (unknown) numerical series. Due to this phenomenon [3], no one was able so far to calculate the pressure at order g^6 or the gluon self-energy at g^4 – a shame for analytical theoretical physics. Today, however, there is a way out. ...

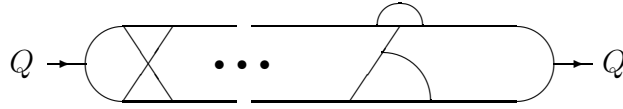


Figure 1: An arbitrary 2-leg n -loop skeleton diagram with one line added: the half circle on top, say, or, equivalently, the one below. The outer momentum Q is static ($Q_0 = 0$) and supersoft ($q \sim m$).

The “Linde sea” of diagrams is easily understood from figure 1. If one more line is added to an arbitrary skeleton diagram, e.g. in the manner shown in the figure, then, in the sense of power counting, it has two more 3-vertices ($\sim p^2 g^2$), three more propagators ($\sim (p^2 + m^2)^{-3}$) and one more loop integration ($T \int d^3 p$, if reduced to the term with zero Matsubara frequency). Thus, the $(n+1)$ -loop and n -loop differ by a factor $\sim g^2 T \int d^3 p p^2 (p^2 + m^2)^{-3} \sim g^2 T / m$. For $m \sim g^2 T$ this factor has order 1 in magnitude. Once the zero-frequency modes become relevant, *all* skeletons contribute with the same order of magnitude. Any finite- n -loop calculation of the magnetic mass is thus inconsistent.

Bosonic fields live on a cylinder with circumference $\beta = 1/T$. Each loop integration $\sum_P \equiv T \sum_n \int_p^3$, $\int_p^3 \equiv (2\pi)^{-3} \int d^3 p$, has its zero-frequency part $T \int_p^3$. A field depending on P loses dependence on its time coordinate in this part. Irrespective of the physical quantity under study, the subset of contributions with $P_0 = 0$ in *all* loops might be the full set of an Euclidean physics at $T = 0$ in three dimensions. However, this theory needs regulators to be derived from the underlying 4D setup. ...

18.2 A few remarks

1. All about the 3D Yang–Mills system is in its functional Hamiltonian \mathbf{H} , (13.23), and the associated scalar product $\langle 1|2 \rangle$, (14.1), involving e^{2NS} . Work on this structure remains to be done. Before all, however, the KKN setup is waiting for quantities to be calculated, because they are relevant in 4D reality.
2. 4D. The hot gas of deconfined interacting gluons. Given a physical quantity of interest, the pressure p , say, or the frequency ω of running waves with wavevector \vec{q} (longitudinal or transverse). This quantity has a prefactor (carrying the dimension, T^4 in p) times a dimensionless function f of the dimensionless coupling g . g is small. By the term *asymptotic expansion* we mean a splitting of f into terms of decreasing order of magnitude — but not necessarily in powers of g . Each such term can in principle be measured. It is thus a physical quantity separately, hence gauge–fixing independent.
3. All diagrams that contribute to a term of the asymptotic expansion (as specified above) must be found and taken into account. It is just this type of consistency we had to learn (and to follow up in the sequel) from the Braaten–Pisarski analysis in 1990. A gauge–fixing independent (gfi) subset may be found, of course, to consist of several gfi subsubsets. In particular, this may happen by decomposing the Matsubara sum into zero– and other modes.
4. Once a Matsubara zero–mode is part of your subsubset, then, for consistency, all diagrams with the same number of external legs, and with the zero–mode selected in each loop, have to be included: the Linde sea (for the quantity at hand). Note that, for the power counting of § 18.1, the number of external legs was irrelevant.
5. If, in general covariant gauges, the gluon propagator is reduced to its transverse part plus the one with gauge–fixing parameter in front of, then the Linde sea diagrams are precisely those of Euclidean YM_3 [3]. This is a superrenormalizable theory. It needs regulators to keep the mass finite. Since being a substructure of 4D, these regulators are to be found in the 4D embedding, namely by watching the other parts of the gluon propagator, see [18, 29]. Working this way, the Linde sea turns into a gfi subset, because now it has become a physics by itself (though in a “wrong” dimension).
6. Such a Linde sea, now representing a physical quantity (or being part of its asymptotic expansion), is a suitably question to the KKN theory. Rather avoid asking KKN “for the exact gluon propagator in covariant gauges”. The latter is gf–dependent, even in the self–energies (except at pole). So, the answer could be: ask physically, you have not yet done your job.
7. Pressure p of the gluon plasma. The zeroth approximation to p is that of 4D free Bosons, hence $\sim T^4$, while the infinity of equal–order diagrams occurs at $g^6 T^4$. The dimensional reduction analysis of Braaten and Nieto [26] ends up at the supersoft scale by addressing some reduced free energy to lattice work. No. Today, this question is to

KKN.

8. Static magnetic screening. The zeroth approximation to the dispersion lines of the gluon plasma is the Braaten–Pisarski setup, hence $\sim g^2 T^2$ in the self-energy. This gives the Debye mass, but zero for the magnetic one. Here, the Linde sea occurs at order $g^4 T^2$. There is no other subsubset, as shown in [18]. Therefore, using $e^2 = g^2 T$, the magnetic screening mass agrees with the *propagator mass*. Arguments, why the latter could be just m , are collected in the next, last subsection.

9. Real photon production. This is one of the LAPTH domains. An exciting detail is found in [27], namely the power counting in equation (1) there. Suitably generalized, it might mean that the zeroth approximation of the production rate of real photons is a Linde sea problem immediately.

Annecy–Le–Vieux, 26. June 2000

18.3 A speculative way out

Apparently, we runned into some conflict. On one hand, at the end of § 15.3, the true YM_{2+1} spectrum was expected to start with white glue balls. Hence, colored one–gluon–states do not exist. On the other hand, 4D TFT needs information about a diagrammatic subset summed up exactly. So, in particular, TFT claims for an exact **one**–gluon spectrum — which does not exist. A disaster ?

The 2+1 D theory can be expanded diagrammatically as well, thereby exhibiting the perfect agreement with the Linde sea. Perturbation theory, even if summed up, must not agree with truth (e.g. $e^{-1/e^2} \neq 0 + 0 + 0 + \dots$). Glue balls are bound states. As such, they are outside the realm of diagrammatics. The glue ball mass m_\bullet cannot be read off from a two–gluon propagator, or in other words, one can n o t learn about the magnetic mass from the true YM_{2+1} spectrum (how to rotate a glue ball into the Euclidean?).

YM_{2+1} has two faces. We might them separate :

The perturbative range

Colored objects exist, propagate and have mass m . Rotation into the Euclidean is possible, as is identification with corresponding subsets of 4D TFT. The whole Linde sea is on this side. Its summation reads

$$m^2 = \frac{1}{2} \text{Tr} \left(A \sum \left[\begin{array}{l} \text{all 2PI self-energy} \\ \text{skeleton diagrams with} \\ (m^2 + q^2)^{-1} \text{ in each line} \end{array} \right] \right)$$

The YM_{2+1} physical reality

The energetically lowest states are white glue balls. Their construction as eigenstates of \mathbf{H} remains a challenge for future time. Vibrations and interaction of balls might complicate the spectrum only at higher energies. Colored objects do not exist. Their energy has turned to infinity.

(18.1)

By the above separation (ignoring the details in (18.1) for a moment), we are led to the next question, namely, how to separate the two regions, or, how to make shure working inside one half. For being on the right side, one just has to work with the full Hamiltonian \mathbf{H} . For being on the left side, “avoid bound state formation”, and “keep colored states at

finite energy". But how ? For the present, with shaking knees, we escape into a

conjecture : to reach the (full) perturbative range
just omit the NT (which is the nonlinear f^{cab} term in \mathbf{H}). (18.2)

We have two arguments (both a bit weak) in support of this conjecture. Nair [19] calls J^a the *gauge-invariant definition of the gluon*. Consider the propagator made up of two such fields (as if they could propagate). In Schroedinger picture language it might read

$$\begin{aligned} i G^{ab} = & \int dt e^{ik_0 t} \int d^2 r e^{-i\vec{k}\vec{r}} \left(\left[\int d\mu(\mathcal{C}) (e^P J^a(\vec{r}))^* e^{-i\mathbf{H}t} e^P J^b(\vec{0}) \right] \theta(t) + \right. \\ & \left. + \left[\int d\mu(\mathcal{C}) (e^P J^b(\vec{0}))^* e^{-i\mathbf{H}t} e^P J^a(\vec{r}) \right] \theta(-t) \right) , \quad \left[\int d\mu(\mathcal{C}) e^{2P} \tilde{J}^{a*}(\vec{q}) e^{-i\tilde{H}t} \tilde{J}^b(\vec{p}) \right] , \end{aligned} \quad (18.3)$$

where $\mathbf{H} e^P = e^P \tilde{H}$ led to the second line. Without NT, we have $\tilde{H} \tilde{J}^b(\vec{p}) = \sqrt{m^2 + p^2} \tilde{J}^b(\vec{p})$ from (15.27) and thus $G^{ab} \sim \delta^{ab}/(k_0^2 - m^2 - k^2)$. With NT, however, G^{ab} has probably no poles (perhaps, this can be cleared up by calculation). In passing, rotation $k_0 \rightarrow iq_3$ makes $-k_0^2 + \vec{k}^2$ to become \vec{q}^2 as it occurs in (18.1).

The second argument arises from a calculation, which includes the NT in a first (still inconsistent) way in the eigenvalue equation. For brevity, let us report it as if it were some (lengthy) exercise. Determine the weight C in the cubic term $P_{\text{cub}} = \int_q \int_p \int_o C^{abc}(\vec{q}, \vec{p}, \vec{o}) \tilde{I}^a(\vec{q}) \tilde{I}^b(\vec{p}) \tilde{I}^c(\vec{o})$ of P by comparing the I^3 terms in (15.5). Obtain $C \sim \{\mathbf{34}\}$. Use $P = P_{\text{qu}} + P_{\text{cub}}$ to get $\tilde{H} = e^{-P} \mathbf{H} e^P$ from (15.19). It contains terms $I\delta$, $I^2\delta$, $I^3\delta$, $\delta\delta$ and $I\delta\delta^9$. Solve $\tilde{H}\chi = E\chi$ by restricting χ , and hence $\tilde{H}\chi$ too, to terms $\sim I$ and $\sim I^2$. This is the announced inconsistency (remember (15.20)). To be specific, $\chi = \tilde{I}^d(\vec{\kappa}) + \int_q \int_p R^{dab}(\vec{\kappa}, \vec{q}, \vec{p}) \tilde{I}^a(\vec{q}) \tilde{I}^b(\vec{p})$. Obtain R from comparing the I^2 terms. Then, equalizing terms $\sim I$, an equation for the eigenvalue E derives. Simplify towards large q , angular integrate and put a cutoff Λ in by hand. One obtains

$$E = \sqrt{m^2 + \kappa^2} + \frac{7}{2} m \int_{m_0}^{\Lambda} dq \frac{1}{2q - E} \quad (18.4)$$

$$\text{with solution} \quad E = \frac{7}{4} m \left[\ln \left(\frac{\Lambda}{m} \right) + O \left(\ln(\ln(\Lambda/m)) \right) \right] , \quad (18.5)$$

showing the irrelevance of the lower cutoff $m_0 = O(m, \kappa)$. By including more and more oscillators (of higher and higher wave vector: $\Lambda \rightarrow \infty$), the energy of this colored state $\chi^d(\vec{\kappa})$ tends to infinity. So, this (first step of a) calculation points in the right direction, as it reveals a dramatic effect of the NT and the removal of colored states, so announced in the right half of (18.1).

There is still all the mystery in the conjecture (18.2), of course. Perhaps, the above provokes anyone to give a better answer.

⁹ Drop the $I^3\delta$ term. Then, \tilde{H} is correct to first order in the NT (as if f^{cab} would be small). But $T_3 R I^2 \sim (\text{NT})^2$ is included in $\tilde{H}\chi = E\chi$. Possibly, (18.4) corresponds to (is part of) a divergent 1-loop self-energy of a ϕ^3 theory, whose renormalization, however, is forbidden.

To end up, let us return to the left half of (18.1), in particular to the magnetic screening mass m_{scr} . It is defined as the position of the pole (at imaginary q [18, 28]) of the transverse piece of the thermal gluon propagator in the static limit $Q_0 = 0$. Pts. 2 to 6 of § 18.2 tell us that m_{scr} is a physical quantity :

$$m_{\text{scr}}^2 = \Pi_t(0, q^2 = -m_{\text{scr}}^2) \quad , \quad \Pi_t := \frac{1}{2} \text{Tr}(\mathbf{A} \Pi) \quad , \quad \mathbf{A} := \frac{\vec{q} \circ \vec{q}}{q^2} - 1 \quad . \quad (18.6)$$

As a 4 by 4 matrix \mathbf{A} has no zeroth components. Note the rotational invariance of Π_t with respect to $\vec{q} = (q_1, q_2, q_3)$. Diagrammatics and some labour concerning possible additional terms and regulators [29] tell us that the function Π_t perfectly agrees with the exact gluon self-energy Π of Euclidean YM_3 , i.e. $\Pi_t(0, q^2) = \Pi(q^2)$. From YM_{2+1} , treated diagrammatically in covariant gauges, one would read off the spectrum k_0 from $0 = k_0^2 - \vec{k}^2 - \Pi(-k_0^2 + \vec{k}^2)$. Here, $\vec{k} = (k_1, k_2)$ is two-component. If one is told the spectrum as $k_0 = \sqrt{m^2 + \vec{k}^2}$ from whatsoever non-perturbative treatment [only here we now make use of the conjecture (18.2)], then $0 = k_0^2 - \vec{k}^2 - \Pi(-k_0^2 + \vec{k}^2)$ turns into

$$m^2 = \Pi(-m^2) \quad \curvearrowright \quad m_{\text{scr}}^2 = m^2 \Big|_{e^2=g^2T} = \frac{g^4 N^2 T^2}{4\pi^2} \quad . \quad (18.7)$$

For the mass identification, compare (18.7) (left) with (18.6) (left). The formula in the left half of (18.1) is the consistency condition for an IR regulator mass [30], still valid when one is forced (by Linde) to become exact. It illustrates that the Linde sea diagrams are summed up, indeed.

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